Control of Piecewise-Affine Hybrid Systems – Lecture 1 Control Problems of Piecewise-Affine Hybrid Systems

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Outline

Outline

Motivation

Motivation for control of hybrid systems

In engineering, control computers are used to implement controllers.
Interaction of continuous and of discrete dynamics
is too tight to ignore for control synthesis.

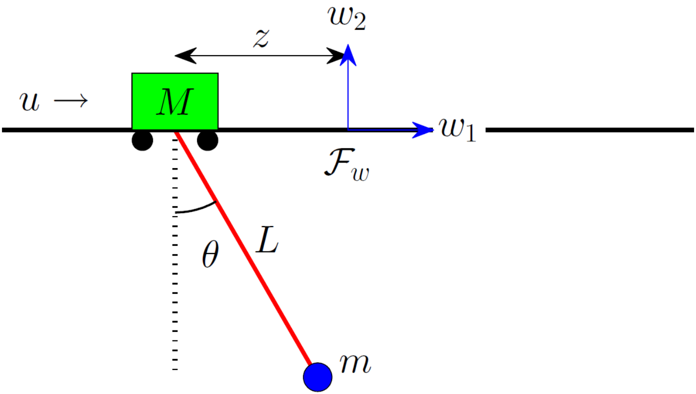
Examples

1. Control of the idle speed of a car engine
   (Parades, CWI; 2002-2004).
2. Conveyor belts (TUE, CWI).
3. Batch chemical plants (TUE).
4. Control of a gantry crane.
5. Control of a robot arm.

Motivation

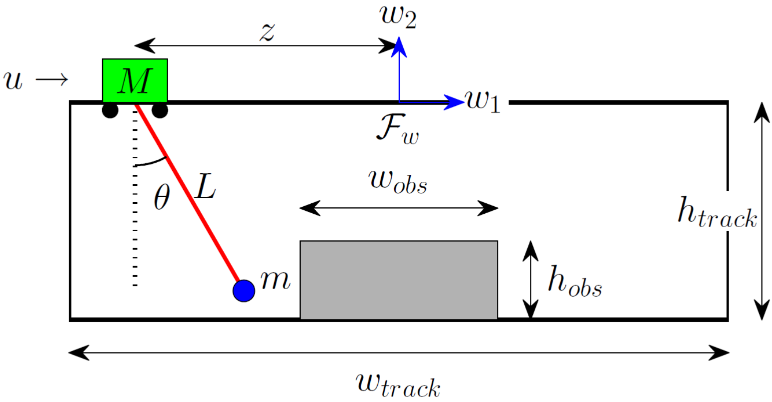
Example. PAHSP – Figure

Source: cdc2014-p3609-3914 (M. Vukslavljev, M.E. Broucke).



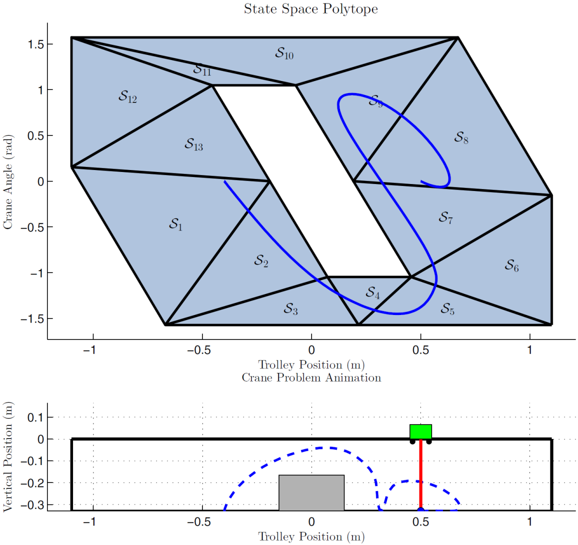
Motivation

Example. PAHSP – Figure



Motivation

Example. PAHSP – Figure



Outline

Linear Systems

Def. Linear control system

Define (time-invariant continuous-time) linear control system
is a control system of control theory such that,

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t), \;\; x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

$T = [0, \infty)$, time index set,

$n, \; m, \; p \in \mathbb{N}$, respectively dimensions of

$X = \mathbb{R}^n, \; U = \mathbb{R}^m, \; Y = \mathbb{R}^p$,

state space, input space, output space,

$x_0 \in X$, initial state,

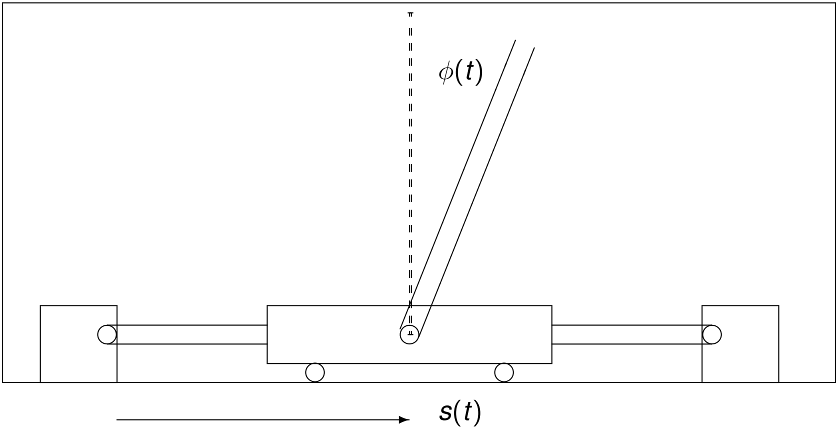
$u : T \to U$, input function,

$x : T \to X$, state function, is solution of ODE,

$y : T \to Y$, output function.

Linear Systems

Example. Inverted pendulum on a cart



Linear Systems

Example. Pendulum on a cart

$$\begin{aligned}
H(t) &= m\frac{d^2}{dt^2}[s(t) + L\sin(\phi(t))], \\
V(t) - mg &= m\frac{d^2}{dt^2}L\cos(\phi(t)), \\
J\frac{d^2\phi(t)}{dt^2} &= LV(t)\sin(\phi(t)) - LH(t)\cos(\phi(t)), \\
M\frac{d^2 s(t)}{dt^2} &= \mu(t) - H(t) - F\frac{ds(t))}{dt}; \\
& \text{approximate, } \ m \ll M; \\
0 &= \frac{d^2\phi(t)}{dt^2} - \frac{g}{L_1}\sin(\phi(t)) + \frac{1}{L_1}\cos(\phi(t))\frac{d^2 s(t)}{dt^2}, \\
& L_1 = \frac{J + mL^2}{mL}; \ \text{approximate, } \sin(\phi) = \phi.
\end{aligned}$$

Linear Systems

Example. Pendulum on a cart

$$
\begin{aligned}
x(t) &= \begin{pmatrix} s(t) & ds(t)/dt & s(t) + L_1\phi(t) & ds(t)/dt + L_1 d\phi(t)/dt \end{pmatrix}^T, \\
u(t) &= \mu(t); \\
dx(t)/dt &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & & 1 \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ b_2 \\ 0 \\ 0 \end{pmatrix} u(t) \\
&= Ax(t) + Bu(t),\ x(0) = x_0; \\
& a_{22} = -F/M,\ a_{42} = -a_{41} = g/L_1,\ b_1 = 1/M, \\
& F/M = 1\ s^{-1},\ 1/M = 1\ kg^{-1}, g/L_1 = 11.65\ s^{-2},\ L_1 = 0.842\ m.
\end{aligned}
$$

H. Kwakernaak, R. Sivan, 1972, Ex. 1.1, pp. 4–7.

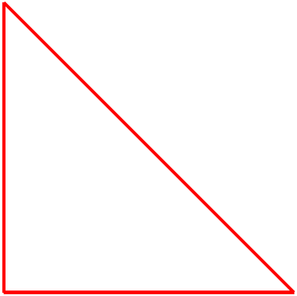
Linear Systems

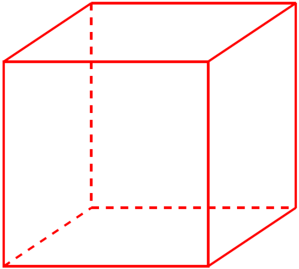
Proposition. Existence state function or state trajectory

$$
\begin{aligned}
& \forall\, x_0 \in X,\ \forall\, u : T \to U,\ \text{piecewise continuous,} \\
& \exists\, x : T \to X,\ \text{differentiable such that,} \\
dx(t)/dt &= Ax(t) + Bu(t),\ \ x(0) = x_0; \\
& \text{define output function or output trajectory,} \\
y(t) &= Cx(t) + Du(t).
\end{aligned}
$$

Outline

Polytopes





Polytopes

Def. Vector space

Vector space V over a field F

$$
\begin{array}{ll}
 & (F, V, 0_F, 1_F, 0_V), \\
 & + : V \times V \to V, \text{ vector addition;} \\
A1 & x + y = y + x, \ \forall\, x, y \in V, \text{ commutativity;} \\
A2 & (x + y) + z = x + (y + z), \ \forall\, x, y, z \in V, \text{ associativity;} \\
A3 & \exists\, 0_V \in V, \ \ x + 0_V = x, \forall\, x \in V; \\
A4 & \forall x \in V, \exists -x \in V, \ x + (-x) = 0_V; \\
 & \times : F \times V \to V, \text{ scalar multiplication;} \\
SM1 & (\alpha\beta) \times x = \alpha \times (\beta \times x), \forall\, \alpha, \beta \in F, \ x \in V; \\
SM2 & \alpha \times (x + y) = \alpha \times x + \alpha \times y, \forall\, \alpha \in F, \ x, y \in V; \\
SM3 & (\alpha + \beta) \times x = \alpha \times x + \beta \times x, \forall\, \alpha, \beta \in F, \ x \in V; \\
SM4 & 1_F \times x = x, \ \forall\, x \in V.
\end{array}
$$

Polytopes

Example. Real vector space

Vector space of n-tuples of the real numbers

$$(\mathbb{R}, \mathbb{R}^n, 0_{\mathbb{R}}, 1_{\mathbb{R}}, 0_{\mathbb{R}^n}),$$

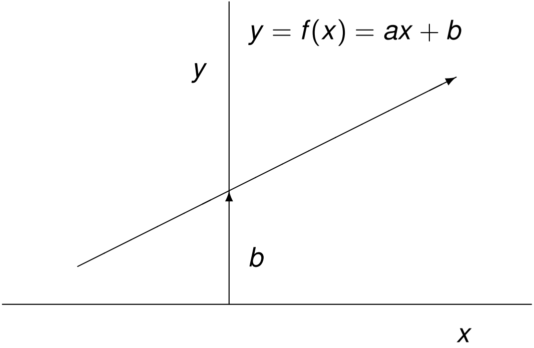
$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3.1 \\ 2.0 \\ 1.2 \end{pmatrix} \in \mathbb{R}^3; \\ x + y &= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}, \quad 7 \times x = \begin{pmatrix} 7x_1 \\ 7x_2 \\ 7x_3 \end{pmatrix}. \end{aligned}$$

Polytopes

Def. Affine function

$$
\begin{aligned}
y &= f(x) = Ax + b, \\
& n, m \in \mathbb{Z}_+,\ X = \mathbb{R}^m,\ Y = \mathbb{R}^n,\ A \in \mathbb{R}^{n \times m},\ b \in \mathbb{R}^n,\ f : X \to Y.
\end{aligned}
$$

Figure. Affine function



Polytopes

Def. Affine subspace

Affine subspace $X \subset \mathbb{R}^n$
if there exists a vector $v_0 \in \mathbb{R}^n$ such that,

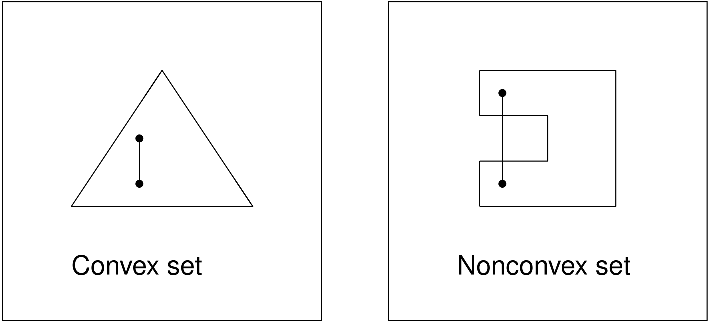
$$X - \{v_0\} = \{x - v_0 \in \mathbb{R}^n | \forall\, x \in X\},$$

is a vector space.
$\mathrm{Affdim}(X) = \dim(X - \{v_0\})$.

$$\begin{aligned} & \{v_0, v_1, \ldots, v_k \in \mathbb{R}^n\}, \ \ k \le n, \ k, n \in \mathbb{Z}_+, \\ & \text{called affinely independent if,} \\ \Leftrightarrow \ \ & \{v_1 - v_0, \ldots, v_k - v_0\}, \\ & \text{are independent vectors in } (\mathbb{R}, \mathbb{R}^n), \\ \Leftrightarrow \ \ & \mathrm{rank}\begin{pmatrix} v_1 - v_0 & \ldots & v_k - v_0 \end{pmatrix} = k. \end{aligned}$$

Polytopes

Figure of a convex set and of a nonconvex set



Polytopes

Def. Convex set

$X \subseteq \mathbb{R}^n$ called a convex set if

$$\forall\, x, y \in X,\ \forall\, \lambda \in [0,1] \subset \mathbb{R},$$
$$[\lambda x + (1-\lambda)y] \in X.$$

Def. Convex hull

Consider a subset $S \subseteq \mathbb{R}^n$.
The convex hull of S is a subset $\mathrm{convh}(S) \subseteq \mathbb{R}^n$
defined as the smallest convex set which contains S.

Polytopes

Def. Polytope

(1) A polytope $P \subset \mathbb{R}^n$ is defined to be
a bounded intersection of a finite number of closed half spaces,

$$\begin{aligned} P &= \cap_{i=1}^{m} H(h_i, c_i), \;\; m \in \mathbb{Z}_+, \; h_i \in \mathbb{R}^n, \; c_i \in \mathbb{R}, \\ H(h_i, c_i) &= \{x \in \mathbb{R}^n | h_i^T x \leq c_i\}, \text{ half space}, \\ HP(h_i, c_i) &= \{x \in \mathbb{R}^n | h_i^T x = c_i\}, \text{ hyperplane.} \end{aligned}$$

(2) A polytope is also the convex hull of a finite number of vectors.

$$\begin{aligned} P &= \mathrm{convh}(\{v_1,\ v_2,\ \ldots,\ v_k\}), \;\; v_i \in \mathbb{R}^n, \; \forall\, i \in \mathbb{Z}_k; \\ V(P) &= \{v_1,\ v_2,\ \ldots,\ v_k\}, \text{ vertex set of } P; \\ & \quad \dim(P) = \dim(\mathrm{affh}(P)) = \dim(\mathrm{span}(v_2 - v_1,\ \ldots,\ v_k - v_1)). \end{aligned}$$

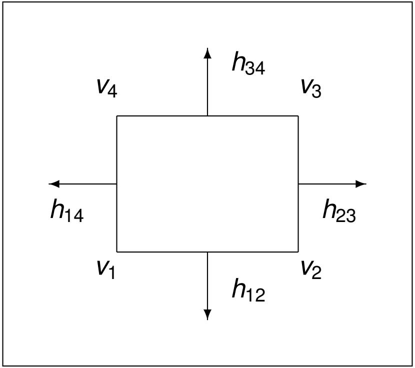
P called full dimensional if $\dim(P) = n$.
$P(\mathbb{R}^n)$ denotes set of polytopes in $\mathbb{R}^n$.
Call (a) implicit form and (b) explicit form of a polytope.

Polytopes

Def. Rectangle

A rectangle is a polytope
if every bounding hyperplane is parallel to the one of the main axes.

Figure of rectangle

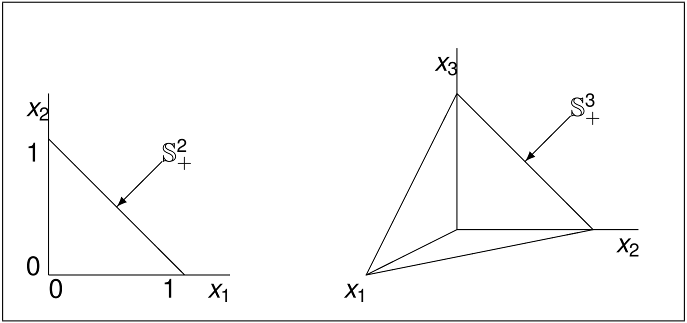


Polytopes

Def. Space of stochastic vectors

$$\mathbb{S}_+^n \quad = \quad \{x \in \mathbb{R}_+^n | 1_n^T x = 1\}, \quad n \in \mathbb{Z}_+; \text{ a polytope.}$$

Figures of sets of stochastic vectors



Polytopes

Proposition. Representation of polytope

$$\begin{aligned} P &= \mathrm{convh}(\{v_1, \ldots, v_k\}), \text{ polytope;} \\ V &= \begin{pmatrix} v_1 & \ldots & v_k \end{pmatrix}, \\ & \forall\, x \in P,\ \exists\, x_s \in \mathbb{S}_+^n, \text{ such that,} \\ x &= Vx_s,\ \ 1_{\mathbb{R}} = 1_n^T x_s. \end{aligned}$$

Example. Nonuniqueness of vertex representation

$$\begin{aligned} P &= \mathrm{convh}(V),\ \ V = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ x \in P \in P(\mathbb{R}^2), \\ x &= \begin{pmatrix} 0.7 \\ 0.8 \end{pmatrix} = V\lambda = V\mu,\ \ 1_4^T\lambda = 1 = 1_4^T\mu, \\ \lambda &= \begin{pmatrix} 0 \\ 0.2 \\ 0.5 \\ 0.3 \end{pmatrix},\ \mu = \begin{pmatrix} 0.2 \\ 0 \\ 0.7 \\ 0.1 \end{pmatrix}, \\ & x \in \mathrm{convh}(\{v_1,\ v_2,\ v_3\}),\ \ x \in \mathrm{convh}(\{v_0,\ v_2,\ v_3\}). \end{aligned}$$

Polytopes

Def. Simplex

A k-dimensional simplex is a polytope such that,

$$\begin{aligned} P &= \operatorname{convh}(\{v_0,\ v_1,\ \ldots,\ v_k\}) \subset \mathbb{R}^n, \\ \dim(P) &= k; \\ V &= \begin{pmatrix} v_0 & v_1 & v_2 & \ldots & v_k \end{pmatrix}, \text{ matrix of vertices.} \end{aligned}$$

Full-dimensional simplex if $\dim(P) = n$.

Canonical form for representation of a full-dimensional simplex,

$$P^n_{cf} = \operatorname{convh}(\{0,\ e_1,\ \ldots, e_n\}) \subset \mathbb{R}^n_+.$$

e_i denotes i-th unit vector of $\mathbb{R}^n$.

Polytopes

Proposition. Unique representation for a simplex

$$
\begin{aligned}
& P \subset \mathbb{R}^n, \text{ full-dimensional simplex } (\mathrm{dim}(P) = n); \\
& \forall\ x \in P,\ \ \exists \text{ unique } x_s \in \mathbb{S}_+^{n+1}, \text{ such that,} \\
x \quad = \quad & V x_s,\ \ 1_{\mathbb{R}} = 1_{n+1}^T x_s\ \ (\Leftrightarrow\ x_s \in \mathbb{S}_+^{n+1}).
\end{aligned}
$$

Proof

Solve the following equation for an unique λ,

$$
\begin{aligned}
\begin{pmatrix} x \\ 1 \end{pmatrix} \quad = \quad & \begin{pmatrix} V \\ 1_{n+1}^T \end{pmatrix} x_s, \quad x_s = \begin{pmatrix} V \\ 1_{n+1}^T \end{pmatrix}^{-1} x; \\
& \mathrm{rank} \begin{pmatrix} V \\ 1_{n+1}^T \end{pmatrix} = \mathrm{rank} \begin{pmatrix} v_0 & v_1 & v_2 & \dots & v_n \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \\
= \quad & \mathrm{rank} \begin{pmatrix} 0 & v_1 - v_0 & v_2 - v_0 & \dots & v_n - v_0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \\
= \quad & 1 + \mathrm{rank} \begin{pmatrix} v_1 - v_0 & v_2 - v_0 & \dots & v_n - v_0 \end{pmatrix} = 1 + n.
\end{aligned}
$$

Polytopes

Example. Unique simplex representation

Consider the simplex and the vector x, and compute the representation,

$$\begin{aligned} P &= \mathrm{convh}(V),\ \ V = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \\ x &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} = V\lambda,\ \ \lambda = \frac{1}{12}\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}. \end{aligned}$$

Polytopes – Polyhedral sets

Def. Polyhedral set

A polyhedral set is defined as
the intersection of a finite number of half spaces.

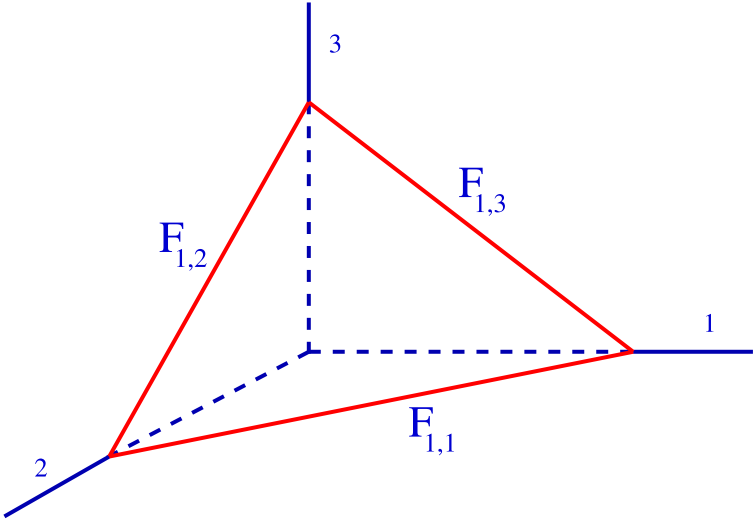
$$\begin{aligned} & \exists\ i_1 \in \mathbb{Z}_+, \text{ such that,} \\ P &= \cap_{i=1}^{i_1} H(h_i, c_i) = \cap\{x \in \mathbb{R}^n | h_i^T x \le c_i\}. \end{aligned}$$

Types of polyhedral sets:
(a) Vector space or an unbounded subset of a vector space. Example.

$$P = [0,2] \times \mathbb{R} = H(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 2) \cap H(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, 0).$$

(b) Cone. Example. $H(\begin{pmatrix} 1 \\ -2 \end{pmatrix}, 0) \cap H(\begin{pmatrix} -3 \\ 1 \end{pmatrix}, 0)$.
(c) A polytope, a bounded polyhedral set. Example. $P = [0,2] \times [0,1]$.

Polytopes – Lattice of Faces



Polytopes – Lattice of Faces

<table>
<tr><td colspan="10">$F_3 = F_{3,(0,1,2,3)}$</td></tr>
<tr><td colspan="6">$F_{2,(1,2,3)}$</td><td>$F_{2,(0,1,3)}$</td><td>$F_{2,(0,2,3)}$</td><td colspan="2">$F_{2,(0,1,2)}$</td></tr>
<tr><td colspan="2">$F_{1,(1,2)}$</td><td colspan="2">$F_{1,(1,3)}$</td><td colspan="2">$F_{1,(2,3)}$</td><td>$\ldots$</td><td></td><td></td><td></td></tr>
<tr><td>$F_{0,1}$</td><td>$F_{0,2}$</td><td>$F_{0,1,}$</td><td>$F_{0,3}$</td><td>$F_{0,2}$</td><td>$F_{0,3}$</td><td></td><td></td><td></td><td></td></tr>
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Polytopes – Lattice of Faces

Def. Face and lattice of faces

Consider polytope $P \subset \mathbb{R}^n$.

Define the lattice of faces recursively by,

$$
\begin{aligned}
& \{F_n(P), F_{n-1}(P), \ldots, F_k(P), \ldots, F_0(P)\}, \\
F_n(P) &= \{F_n\} = \{P\},\ \text{n-th dimensional face of } P, \\
F_{n-1}(P) &= \{F_{n-1,1},\ \ldots,\ F_{n-1,k_{n-1}}\}, \\
& F_{n-1,i} = F_n \cap HP(h_i, c_i),\ \dim(F_{n-1,i}) = n-1,\ \forall\ i \in \mathbb{Z}_{k_{n-1}}; \\
& \partial F_n = \cup_{i=1}^{k} F_{n-1,i},\ \text{boundary of } F_n, \\
& \text{boundary } F_n \text{ has a polyhedral partition;} \\
& F_{n-1,i} \text{ called i-th facet of } P; \\
F_{n-2}(P) & \quad \ldots,\ \text{set of $(n-2)$th dimensional faces of } P, \\
&\vdots \\
F_0(P) &= \{v_0,\ \ldots,\ v_n\} = V(P),\ \text{set of 0-th dimensional faces of } P.
\end{aligned}
$$

Polytopes

Def. Faces of a polytope

Define the vertices of a face of a polytope as,

$$
\begin{aligned}
P &= \mathrm{convh}(\{v_1, \ldots, v_k\}) \subset \mathbb{R}^n, \text{ a polytope,} \\
&\quad F_{k,i} \in F_k(P), \\
V(F_{k,i}) &\subseteq V(P), \text{ such that,} \\
F_{k,i} &= \mathrm{convh}(V(F_{k,i})).
\end{aligned}
$$

Def. Facets of a simplex

Convention.

$$
\begin{aligned}
P &= \mathrm{convh}(\{v_0, v_1, \ldots, v_n\}) \subset \mathbb{R}^n, \text{ a simplex,} \\
F_{n-1,i} &= \mathrm{convh}(\{v_j \in \mathbb{R}^n, \ \forall\, j \in \mathbb{N}_n \backslash \{i\}\}), \\
&\quad h_i \in \mathbb{R}^n, \text{ normal of } F_{n-1,i}, \text{ pointing outward;} \\
F_{n-1,0} &= P \cap HP(h_0, h_0^T v_1) = \{x \in P | h_0^T x = h_0^T v_1\}.
\end{aligned}
$$

Outline

Affine System

Def. Affine control system on a polytope (ACSP)

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a, \; x(t_0) = x_0 \in X_0 \subseteq X, \\
y(t) &= Cx(t) + Du(t) + c, \\
& \quad U \subset \mathbb{R}^m, \; Y \subset \mathbb{R}^p, \text{ polytopes}, \\
& \quad X \subset \mathbb{R}^n, \; \dim(X) = n, \text{ closed full-dim. polytope}, \\
t_1 &= \inf\{t \in T \cup \{+\infty\} | x(t) \in F_{n-1,r}\}, \\
& \quad \text{lifetime of state trajectory in polytope;} \\
T_1 &= [t_0, \infty), \text{ if } t_1 = \infty, \text{ or}, \\
T_1 &= [t_0, t_1] \subset R_+, \text{ if } t_1 < \infty, \text{then } u, x, y \text{ defined on } T_1, \\
F_{n-1,r} & \quad \text{called exit facet if trajectory leaves } X \text{ via this facet.}
\end{aligned}
$$

Affine System

Comments. Affine control system on a polytope

- Affine control system because the system equations,

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a, \ x(t_0) = x_0 \in X_0 \subseteq X, \\ y(t) &= Cx(t) + Du(t) + c, \end{aligned}$$

  are affine in $x(t)$ and in $u(t)$,
  $Ax + (Bu + a)$ and $Bu + (Ax + a)$.
- Why is it necessary that the system is affine? See next slide.
- Input space may be polytope also.
  Example. $U = [0, 1]^m$. Useful to model input constraints.

Affine Systems

Proposition. Transformation of polytope

Consider an affine control system on a polytope.
Apply a state-set transformation of the form,

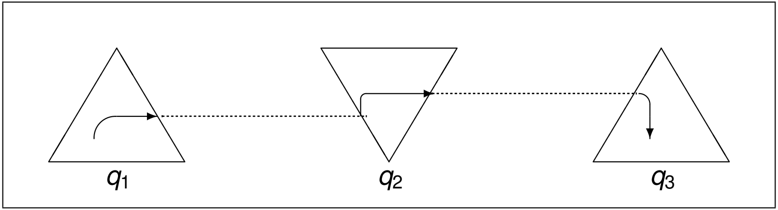
$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a, \; x(t_0) = x_0, \\ \overline{x}(t) &= f(x(t)) = Sx(t) + k, \\ & S \in \mathbb{R}^{n \times n}, \;\; \mathrm{rank}(S) = n, \;\; k \in \mathbb{R}^n, \\ d\overline{x}(t)/dt &= SAS^{-1}\overline{x}(t) + SBu(t) + (Sa - SAS^{-1}k), \\ & (Sa - SAS^{-1}k) \in \mathbb{R}^n. \end{aligned}$$

The constant vector has shifted!

Outline

Control PAHCSP

Figure of PAHCSP



PAHCSP

Def. Piecewise-affine hybrid control system on a polytope
(PAHCSP, continuous time, time-invariant)

$$\begin{array}{rcl}
Q & & \text{finite state set}, \; U \subset \mathbb{R}^m, \; Y \subset \mathbb{R}^p, \; \text{polyhedral sets}, \\
X_q & \subset & \mathbb{R}^{n_q}, \; \forall q \in Q, \; \text{closed polyhedral sets}, \\
dx_q(t)/dt & = & A(q)x_q(t) + B(q)u(t) + a(q), \; x_q(t_o) = x_q^+, \\
y(t) & = & C(q)x_q(t) + D(q)u(t) + c(q), \\
e & \in & E_{in}, \; \text{input event, or} \\
e & \in & E_{cd}, \; \text{if } x(t_1) \in G_q(e) \subset \partial X_q, \; \text{guard}; \\
 & & \text{event generated by continuous dynamics; then transition,} \\
q^+ & = & f(q^-, x_{q^-}^-, e), \; q_0, \\
x_{q^+}^+ & = & A_r(q^-, e, q^+)x_{q^-}^- + b_r(q^-, e, q^+), \; \text{reset map}.
\end{array}$$

Assumptions: (1) Finite number of events at any time.
(2) Finite number of events on any finite interval (non-Zenoness).

Choice of PAHCSP

Remarks

- Examples mentioned belong to class PAHCSP.
- In hybrid systems, modeling of events generated by continuous dynamics. Event occurs when boundary of state set is reached.
- Polytope as state set, defined by a finite set of inequalities.
- PAHSP generalizes timed automata and automata on rectangles analyzed by computer scientists.
- PAHSP class of hybrid systems proposed by E.D. Sontag (1981).
- PAHS related to:
  - Mixed Logic Dynamical Systems [A. Bemporad, M. Morari] (DT only) and to
  - Linear Complementarity Systems [M. Heemels].
- Affine control systems useful simple class.
- System theory - Proper choice of class of dynamic control systems.

Outline

Control Theory

Def. Control law

Consider a linear control system,

$$dx(t)/dt \quad = \quad Ax(t) + Bu(t),\; x(0) = x_0.$$

Define a control law as a function, $g : X \to U$,
based on complete observations.

Def. Types of control laws

- Affine control law
  $g(x) = Fx + r$, $F \in \mathbb{R}^{m \times n}$ and $r \in \mathbb{R}^m$.
- Piecewise-affine control law.
- Continuous piecewise-affine control law.
- Continuous control law.

Control Theory

Def. Closed-loop system

Define the closed-loop system associated with
a control system and a control law by the equations,

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bg(x(t)),\ x(0) = x_0, \\
u(t) &= g(x(t)), \\
dx(t)/dt &= Ax(t) + BFx(t) = (A + BF)x(t),\ x(0) = x_0, \\
u(t) &= Fx(t).
\end{aligned}
$$

For a nonlinear control law one needs to assume
existence of a solution to the nonlinear differential equation.
May have to a use a Filippov solution for the closed-loop system.

Remark. Control law vs input

Distinguish: (1) control law $g : X \to U$,
(2) input function $u : T \to U$.

Control Theory

Def. Control objectives

Define control objectives for the closed-loop system:

- State transfer:
  Transfer the system from an initial state to a terminal state.
- Stability: Keep the system at or close to a steady state even when there are unforeseen input disturbances.
- Disturbance rejection:
  Eliminate or diminish the effect of a disturbance or noise.
- Performance optimization:
  Optimize a particular performance criterion over a time interval.
- Robustness: Maintain stability and performance
  if the system is different from the assumptions,
  or if it changes slightly over time.
- Adaptive control:
  Adapt the inputs to small changes in the system or
  to slow changing external input signals.

Control Theory

Control synthesis

Determine a control law or a subclass of control laws
which meets one or more of the specified control objectives.

Example. Control synthesis of a linear system

$$dx(t)/dt \quad = \quad Ax(t) + Bu(t),\ x(0) = x_0.$$

Determine a subclass of the class of linear control laws $G_1 \subseteq G_L$
which optimize the performance of a quadratic cost function.

Control design

Determine the parameter values of a control law which achieve optimal performance for a particular control objective.

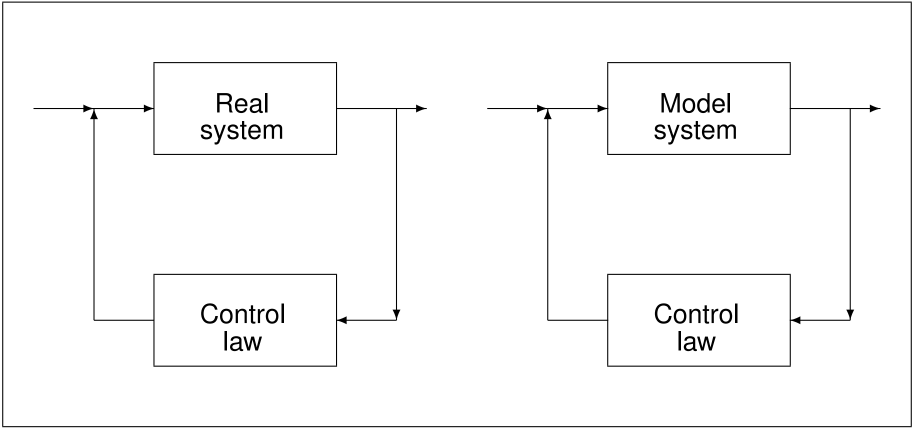
Example. Control design for stability

Determine the parameter values of a linear control law of which the closed-loop system is asymptotically stable.

$$F \in \mathbb{R}^{m \times n},\ \ \mathrm{spec}(A + BF) \subset \mathbb{C}^- \ (\Leftrightarrow\ \Re(\lambda(A + BF)) < 0).$$

Control Theory

Real System and Control Law



Outline

Control ACSS

Problem. Control-to-facet 1

Consider an affine control system on a polytope,

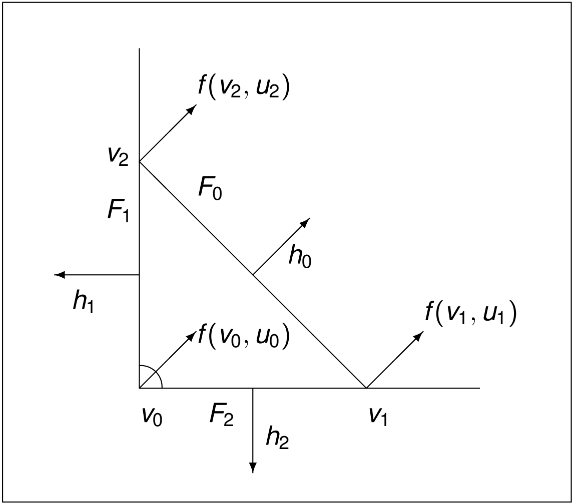
$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a = f(x(t), u(t)),\ x(t_0) = x_0, \\ & X = P_n \subset \mathbb{R}^n,\ \text{a closed full-dim. polytope}, \\ & U \subset \mathbb{R}^m,\ \text{a closed polyhedral set}, \\ & F_0 = F_{n-1,0} \subset P,\ \text{a facet}. \end{aligned}$$

For all $x_0 \in X$ determine
a terminal time $t_1 \in [t_0, \infty)$ and an input $u : [t_0, t_1] \to U$
such that the state trajectory reaches facet F_0:

1. $\forall t \in [t_0, t_1]$, $x(t) \in X = P_n$;
2. $x(t_1) \in F_0$, $t_1 \in T$ smallest such time;
3. $h_0^T \frac{dx(t)}{dt}|_{t=t_1} > 0$, normal vector of F_0 points outward.

Preferably by a continuous control law.

Control ACSP



Control ACSP

Remark. Notation closed-loop system

Consider

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0,\ \text{ ACSS system,} \\
g(x) &= Fx + r,\ \text{affine control law,} \\
dx(t)/dt &= (A + BF)x(t) + (a + Br),\ x(t_0) = x_0,\ \text{closed-loop system,} \\
u(t) &= g(x(t)) = Fx(t) + r,\ \text{input function.}
\end{aligned}
$$

Control ACSP

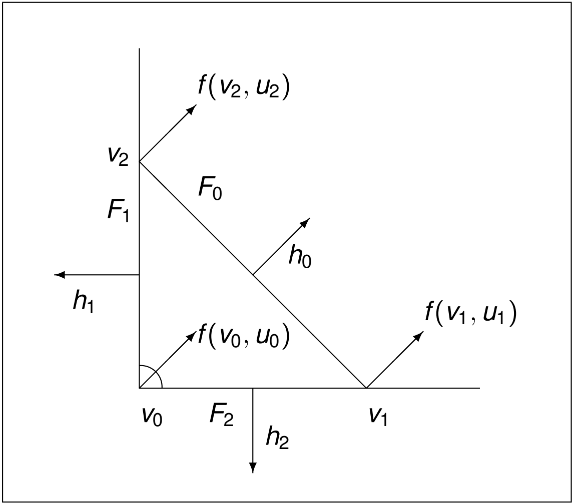
Remark

Attempt-to-exit defined as,

$$
\begin{aligned}
x_1 &= x(t_1), \\
& \frac{dx(t)}{dt}|_{x=x_1} = f(x_1) = Ax_1 + Bu_1 + a, \\
& h_0^T[f(x_1) - x_1] > 0 \; \forall \; x_1 \in F_0, \\
\Leftrightarrow &\quad h_0^T f(x_1) > h_0^T x_1 \; \forall \; x_1 \in F_0; \\
\Leftrightarrow &\quad h_0^T f(v_j) > h_0^T v_j, \;\; \forall \; v_j \in V(F_0).
\end{aligned}
$$

See figure next slide.

Control ACSP



Control ACSP

Problem. Control-to-facet 2

Consider an affine control system on a polytope,

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\
& X \subset \mathbb{R}^n,\ \text{a closed full-dim. polytope}, \\
& U \subset \mathbb{R}^m, \text{a closed polyhedral set}, \\
& F_0 = F_{n-1,0} \subset P,\ \text{a facet.}
\end{aligned}
$$

For all $x_0 \in X$ determine
a terminal time $t_1 \in [t_0, \infty)$ and an input $u : [t_0, t_1] \to U$
such that the state trajectory reaches facet F_0:

1. $\forall t \in [t_0, t_1],\ x(t) \in X = P_n$;
2. $x(t_1) \in F_0$;
3. $h_0^T \frac{dx(t)}{dt}|_{t=t_1} > 0$.

Preferably by a continuous control law g such that $u(t) = g(x(t))$.

Control ACSP

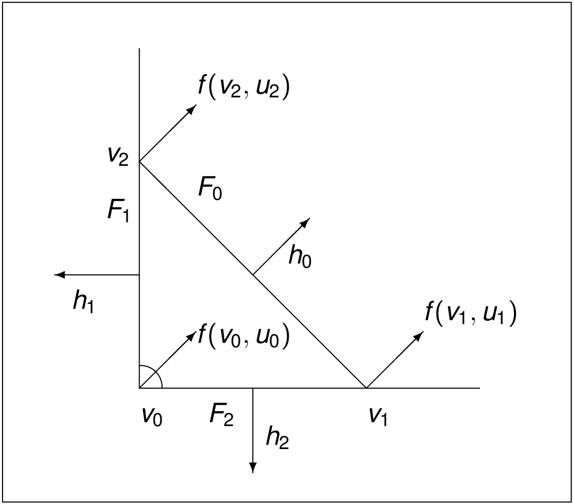
Proposition. Necessary conditions

State set is polytope $X = \mathrm{convh}(\{v_1, \ldots, v_M\}) \subset R^N$.
If there exists a continuous control law $f : X \to U$
for Problem 'Control-to-facet' with exit facet F_1
then there exists $u_1, u_2, \ldots, u_M \in U$ such that

$$\begin{array}{lll}
(1) \quad \forall j \in V(F_1): & (1.a) & h_1^T(Av_j + Bu_j + a) > 0; \\
& (1.b) & \forall i \in F(v_j)\backslash\{1\}: \; h_i^T(Av_j + Bu_j + a) \leq 0; \\
(2) \quad \forall j \in \mathbb{Z}_M\backslash V(F_1): & (2.a) & \forall i \in F(v_j): \; h_i^T(Av_j + Bu_j + a) \leq 0; \\
& (2.b) & \sum_{i \in F(v_j)} h_i^T(Av_j + Bu_j + a) < 0.
\end{array}$$

Note, inequalities describe that vectors at the vertices of the polytope of the closed-loop system be in specified polyhedral cone.
Proof outline. Define $u_j = f(v_j)$.
(automatica-v40p21-35, Proposition 3.1).

Control ACSP



Control ACSP

Theorem. Sufficient condition – Simplex case

Case $X \subset \mathbb{R}^n$ full-dimensional simplex, $\dim(X) = n$, and $F_0 \in F_{n-1}(X)$.

$$dx(t)/dt = Ax(t) + Bu(t) + a,\ x(t_0) = x_0.$$

Assume necessary conditions (1.a,1.b,2.a,2.b) hold
for $u_0, \ldots, u_n \in U$ and F_0. Define

$$f(x) = \sum_{j=1}^{n+1} \lambda_i u_i, \text{ if } x = \sum_{j=1}^{n+1} \lambda_i v_i,$$
$$f : X \to U, \text{ affine control law.}$$

Then f is an affine control law and
a solution to Problem ‘Control-to-facet 2’ for exit facet F_0.
(automatica-v40p21-35, Th. 4.2).

Control ACSP

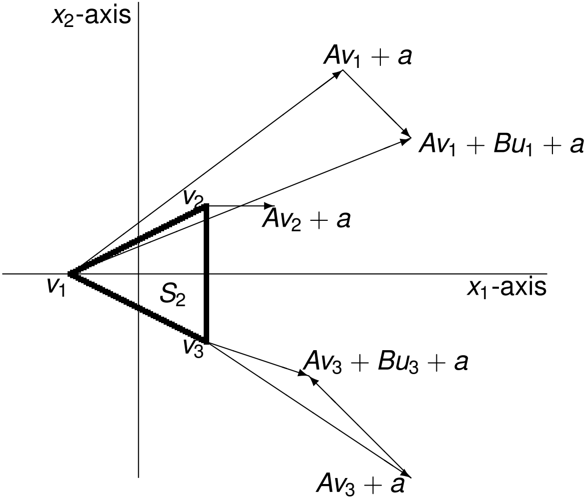


Figure: Control of the vector $\dot{x}$ at the vertices of S_2

Control ACSP

Proof outline

- Conditions imply that at vertices the drift vector points in the objective directions.
- Because of convexity,
  on exit facet, drift vectors point out off the polytope,
  on other facets, drift vectors point to inside of polytope.
- From any point in polytope, exit facet is reached in finite time.

□

Control ACSP

How to determine the input vectors?

Computations decompose per vertex. Simplex case.

$$
\begin{aligned}
& X = P \in P(\mathbb{R}^2),\ U = \mathbb{R}^2 \text{ or } U \in P(\mathbb{R}^2), \\
& \text{compute, } \{u_j \in U,\ j \in V(P)\}, \\
& \forall\ v_j \in V(F_0) = \{v_r, r \in V(P) \backslash \{0\}\}, \\
& h_k^T[Av_j + Bu_j + a - v_j] \leq 0,\ \forall\ F_{n-1,k} \in F_{n-1}(X),\ v_j \in F_{n-1,k}, \\
& h_0^T[Av_j + Bu_j + a - v_j] > 0; \\
& v_0 \in V(X), \\
& h_k^T[Av_0 + Bu_0 + a - v_0] \leq 0,\ \forall\ F_{n-1,k} \in F_{n-1}(X),\ v_0 \in F_{n-1,k}; \\
& Av_0 + Bu_0 + a - v_0 \neq 0.
\end{aligned}
$$

There exists software to determine
the existence of u_j vectors and of u_0 vector,

Control ACSP

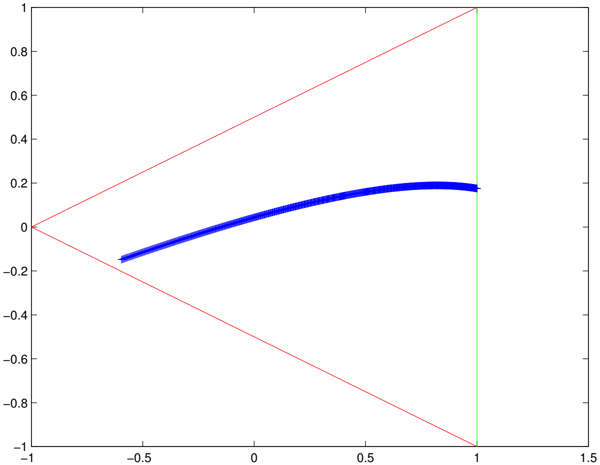
Procedure. Computation control law

1. Solve set of linear inequalites (1.a, 1.b, 2.a, 2.b) for $u_1, \ldots, u_{n+1} \in U$.
   Software for linear inequalities:
   - CDD Library (Fukuda (ETHZ)).
   - New Polka Library (B. Jeannet (Verimag)).
2. Solve the following linear equation for $F \in \mathbb{R}^{m \times N},\ r \in \mathbb{R}^m$,

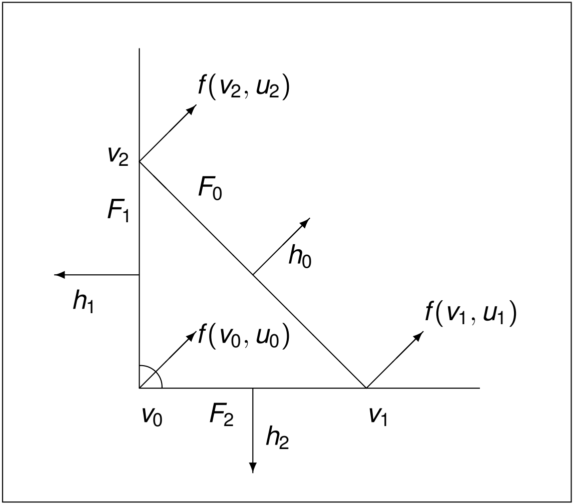
$$\begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_{n+1}^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ r^T \end{pmatrix} = \begin{pmatrix} u_1^T \\ \vdots \\ u_{n+1}^T \end{pmatrix}.$$

   Matrix with v's is nonsingular because simplex is full-dimensional.
3. Control law is $g(x) = Fx + r$.

Control ACSP



Control ACSP



Control ACSP

Def. Vectors at the vertices (VV)

$$
\begin{aligned}
VV_s(v_i) &= \{(Av_i + Bu + a) - v_i \in \mathbb{R}^n | \forall\ u \in U\}, \\
& \quad \text{a cone or a polyhedral set depending on } U, \\
VV_{co}(v_i) &= \{(Av_i + Bu + a) - v_i \in \mathbb{R}^n | \exists\ u \in U,\ \text{inequalities below all hold}\}, \\
& \quad h_j^T[(Av_i + Bu + a) - v_i] \leq 0,\ h_0^T[(Av_i + Bu + a) - v_i] > 0, \\
& \quad\quad \forall\ F_j \in F_{n-1}(X) \backslash \{F_0\},\ v_i \in V(F_0); \\
VV_{co}(v_0) & \quad h_j^T[(Av_0 + Bu + a) - v_0] \leq h_j^T v_0,\ \forall\ F_j \in F_{n-1}(X),\ v_0 \in V(F_j); \\
& \quad \exists\ F_j \in F(X) \backslash \{F_0\},\ h_j^T[(Av_0 + Bu + a) - v_0] > 0, \\
&\Leftarrow \sum_{v_j \in V(F_0)} h_j^T[(Av_0 + Bu + a) - v_0] > 0; \\
& \quad VV_s(v_i) \cap VV_{co}(v_i) \neq \emptyset,\ \ \forall\ \in \mathbb{N}_n.
\end{aligned}
$$

Interpretation of controllability conditions, latter inequality, equivalent condition for satisfaction of control objectives.

Control ACSP

Def. Sufficient conditions for exiting facet F_0

$$
\begin{array}{lll}
(1) & & VV_s(v_0) \cap VV_{co}(v_0) \neq \emptyset,\ v_0 \in X = S(\mathbb{R}^n), \\
 & \Leftrightarrow & (\forall\ F_{n-1,j} \in F_{n-1}(X),\ h_j^T[Av_0 + Bu_0 + a - v_0] \leq 0); \\
(2) & & (\forall\ v_i \in V(F_0),\ VV_s(v_i) \cap VV_{co}(v_i) \neq \emptyset), \\
 & \Leftrightarrow & \left\{ \begin{array}{l} h_j^T[Av_i + Bu_i + a - v_i] \leq 0, \\ h_0^T[Av_i + Bu_i + a - v_i] > 0, \end{array} \right. ; \\
(3) & & \text{no steady state in } X, \\
 & \Leftarrow & [Av_0 + Bu_0 + a - v_0] \neq 0,\ \ \text{related to (1).}
\end{array}
$$

Control ACSP

Proposition. Time to exit

Upper bound on travel time from initial state to exit is,

$$
\begin{aligned}
& F_{n-1,1} \in F_{n-1}(X), \text{ exit facet, normal vector } h_1 \in \mathbb{R}^n, \\
& t_1 \leq [\beta - \alpha]/c_1, \\
\beta &= \max_{j \in \mathbb{Z}_{n_v}} h_1^T v_j, \quad \alpha = \min_{j \in \mathbb{Z}_{n_v}} h_1^T v_j, \\
c_1 &= \min_{j \in \mathbb{Z}_{n_v}} h_1^T [A v_j + B u_j + a].
\end{aligned}
$$

(automatica-v40p21-35, p. 29).

Input space dimension

If $m = n$ and $\mathrm{rank}(B) = n$
then the controllability conditions for the vectors are always satisfied.