Control of Piecewise-Affine Hybrid Systems – Lecture 2 Control Synthesis in Case of Complete Observations

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Outline

Outline

Control ACSS

Problem. Control-to-facet 1

Consider an affine control system on a polytope,

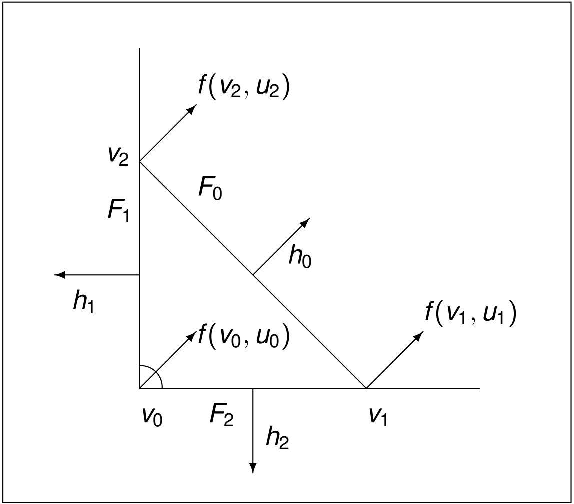
$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a = f(x(t), u(t)),\ x(t_0) = x_0, \\ & X = P_n \subset \mathbb{R}^n,\ \text{a closed full-dim. polytope}, \\ & U \subset \mathbb{R}^m,\ \text{a closed polyhedral set}, \\ & F_0 = F_{n-1,0} \subset P,\ \text{a facet}. \end{aligned}$$

For all $x_0 \in X$ determine
a terminal time $t_1 \in [t_0, \infty)$ and an input $u : [t_0, t_1] \to U$
such that the state trajectory reaches facet F_0:

1. $\forall t \in [t_0, t_1]$, $x(t) \in X = P_n$;
2. $x(t_1) \in F_0$, $t_1 \in T$ smallest such time;
3. $h_0^T \frac{dx(t)}{dt}|_{t=t_1} > 0$, normal vector of F_0 points outward.

Preferably by a continuous control law.

Control ACSP



Control ACSP

Remark. Notation closed-loop system

Consider

$$\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0,\ \text{ ACSS system,} \\
g(x) &= Fx + r,\ \text{affine control law,} \\
dx(t)/dt &= (A + BF)x(t) + (a + Br),\ x(t_0) = x_0,\ \text{closed-loop system,} \\
u(t) &= g(x(t)) = Fx(t) + r,\ \text{input function.}
\end{aligned}$$

Control ACSP

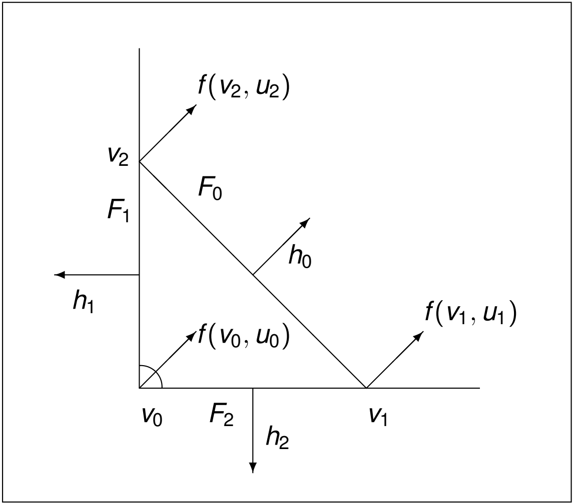
Remark

Attempt-to-exit defined as,

$$\begin{aligned}
x_1 &= x(t_1), \\
& \frac{dx(t)}{dt}\Big|_{x=x_1} = f(x_1) = Ax_1 + Bu_1 + a, \\
& h_0^T[f(x_1) - x_1] > 0 \ \forall \ x_1 \in F_0, \\
\Leftrightarrow & \ h_0^T f(x_1) > h_0^T x_1 \ \forall \ x_1 \in F_0; \\
\Leftrightarrow & \ h_0^T f(v_j) > h_0^T v_j, \ \ \forall \ v_j \in V(F_0).
\end{aligned}$$

See figure next slide.

Control ACSP



Control ACSP

Problem. Control-to-facet 2
Consider an affine control system on a polytope,

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\ & X \subset \mathbb{R}^n,\ \text{a closed full-dim. polytope}, \\ & U \subset \mathbb{R}^m, \text{a closed polyhedral set}, \\ & F_0 = F_{n-1,0} \subset P,\ \text{a facet}. \end{aligned}$$

For all $x_0 \in X$ determine
a terminal time $t_1 \in [t_0, \infty)$ and an input $u : [t_0, t_1] \to U$
such that the state trajectory reaches facet F_0:

1. $\forall t \in [t_0, t_1],\ x(t) \in X = P_n$;
2. $x(t_1) \in F_0$;
3. $h_0^T \frac{dx(t)}{dt}|_{t=t_1} > 0$.

Preferably by a continuous control law g such that $u(t) = g(x(t))$.

Control ACSP

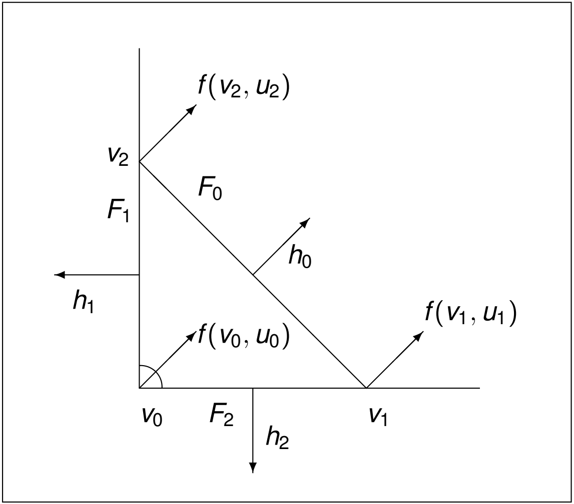
Proposition. Necessary conditions

State set is polytope $X = \mathrm{convh}(\{v_1, \ldots, v_M\}) \subset R^N$.
If there exists a continuous control law $f : X \to U$
for Problem 'Control-to-facet' with exit facet F_1
then there exists $u_1, u_2, \ldots, u_M \in U$ such that

$$
\begin{array}{lll}
(1) & \forall j \in V(F_1): & (1.a)\ \ h_1^T(Av_j + Bu_j + a) > 0; \\
 & & (1.b)\ \ \forall i \in F(v_j)\backslash\{1\}:\ h_i^T(Av_j + Bu_j + a) \leq 0; \\
(2) & \forall j \in \mathbb{Z}_M\backslash V(F_1): & (2.a)\ \ \forall i \in F(v_j):\ h_i^T(Av_j + Bu_j + a) \leq 0; \\
 & & (2.b)\ \ \sum_{i \in F(v_j)} h_i^T(Av_j + Bu_j + a) < 0.
\end{array}
$$

Note, inequalities describe that vectors at the vertices of the polytope of the closed-loop system be in specified polyhedral cone.
Proof outline. Define $u_j = f(v_j)$.
(automatica-v40p21-35, Proposition 3.1).

Control ACSP



Control ACSP

Theorem. Sufficient condition – Simplex case

Case $X \subset \mathbb{R}^n$ full-dimensional simplex, $\dim(X) = n$, and $F_0 \in F_{n-1}(X)$.

$$dx(t)/dt = Ax(t) + Bu(t) + a, \; x(t_0) = x_0.$$

Assume necessary conditions (1.a,1.b,2.a,2.b) hold
for $u_0, \ldots, u_n \in U$ and F_0. Define

$$g(x) = \sum_{j=1}^{n+1} \lambda_i u_i, \text{ if } x = \sum_{j=1}^{n+1} \lambda_i v_i,$$
$$g : X \to U, \text{ affine control law.}$$

Then g is an affine control law and
a solution to Problem ‘Control-to-facet 2’ for exit facet F_0.
(automatica-v40p21-35, Th. 4.2).

Control ACSP

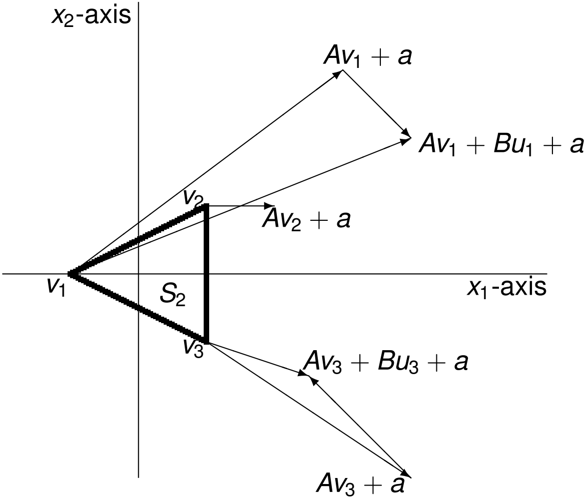


Figure: Control of the vector $\dot{x}$ at the vertices of S_2

Control ACSP

Proof outline

- Conditions imply that at vertices the drift vector points in the objective directions.
- Because of convexity,
  on exit facet, drift vectors point out off the polytope,
  on other facets, drift vectors point to inside of polytope.
- From any point in polytope, exit facet is reached in finite time.

□

Control ACSP

How to determine the input vectors?

Computations decompose per vertex. Simplex case.

$$
\begin{aligned}
& X = P \in P(\mathbb{R}^2),\ U = \mathbb{R}^2 \text{ or } U \in P(\mathbb{R}^2), \\
& \text{compute, } \{u_j \in U,\ j \in V(P)\}, \\
& \forall\ v_j \in V(F_0) = \{v_r, r \in V(P) \backslash \{0\}\}, \\
& h_k^T[Av_j + Bu_j + a - v_j] \leq 0,\ \forall\ F_{n-1,k} \in F_{n-1}(X),\ v_j \in V(F_{n-1,k}), \\
& h_0^T[Av_j + Bu_j + a - v_j] > 0; \\
& v_0 \in V(X), \\
& h_k^T[Av_0 + Bu_0 + a - v_0] \leq 0,\ \forall\ F_{n-1,k} \in F_{n-1}(X),\ v_0 \in V(F_{n-1,k}); \\
& Av_0 + Bu_0 + a - v_0 \neq 0.
\end{aligned}
$$

There exists software to determine
the existence of u_j vectors and of u_0 vector,

Control ACSP

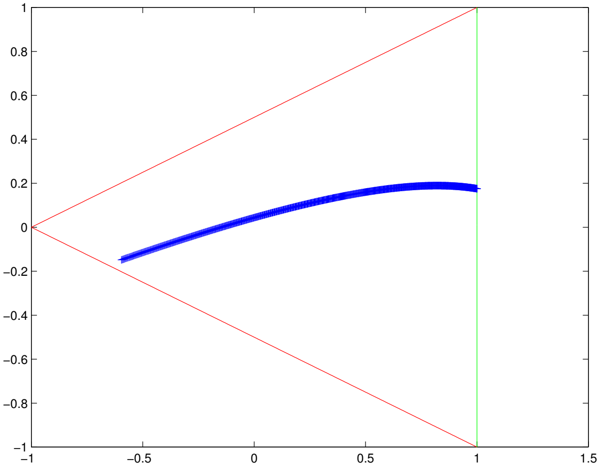
Procedure. Computation control law

1. Solve set of linear inequalites (1.a, 1.b, 2.a, 2.b)
   for $u_1, \ldots, u_{n+1} \in U$.
   Software for linear inequalities:
   - CDD Library (Fukuda (ETHZ)).
   - New Polka Library (B. Jeannet (Verimag)).
2. Solve the following linear equation for $F \in \mathbb{R}^{m \times N},\ r \in \mathbb{R}^m$,

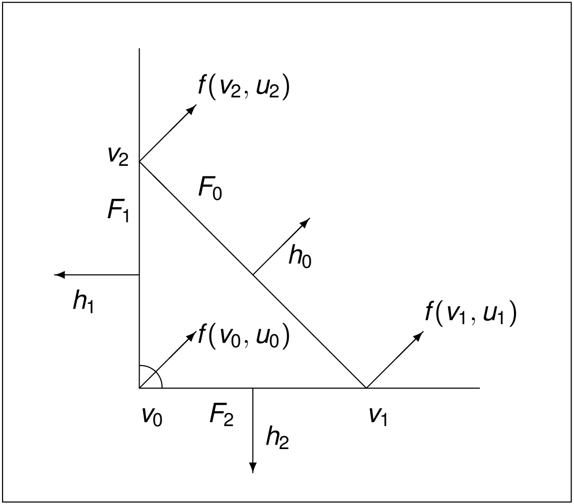
$$\begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_{n+1}^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ r^T \end{pmatrix} = \begin{pmatrix} u_1^T \\ \vdots \\ u_{n+1}^T \end{pmatrix}.$$

   Matrix with v's is nonsingular because simplex is full-dimensional.
3. Control law is $g(x) = Fx + r$.

Control ACSP



Control ACSP



Control ACSP

Def. Vectors at the vertices (VV)

$$
\begin{aligned}
VV_s(v_i) \;&=\; \{(Av_i + Bu + a) - v_i \in \mathbb{R}^n | \forall\ u \in U\}, \\
&\quad\ \text{a cone or a polyhedral set depending on } U, \\
VV_{co}(v_i) \;&=\; \{(Av_i + Bu + a) - v_i \in \mathbb{R}^n | \exists\ u \in U,\ \text{inequalities below all hold}\}, \\
&\quad\ h_j^T[(Av_i + Bu + a) - v_i] \le 0,\ h_0^T[(Av_i + Bu + a) - v_i] > 0, \\
&\quad\quad \forall\ F_j \in F_{n-1}(X)\backslash\{F_0\},\ v_i \in V(F_0); \\
VV_{co}(v_0) \;&\quad\ h_j^T[(Av_0 + Bu + a) - v_0] \le h_j^T v_0,\ \forall\ F_j \in F_{n-1}(X),\ v_0 \in V(F_j); \\
&\quad\ \exists\ F_j \in F(X)\backslash\{F_0\},\ h_j^T[(Av_0 + Bu + a) - v_0] > 0, \\
\Leftarrow &\quad \sum_{v_j \in V(F_0)} h_j^T[(Av_0 + Bu + a) - v_0] > 0; \\
&\quad\ VV_s(v_i) \cap VV_{co}(v_i) \neq \emptyset,\ \ \forall\ \in \mathbb{N}_n.
\end{aligned}
$$

Interpretation of controllability conditions, latter inequality, equivalent condition for satisfaction of control objectives.

Control ACSP

Def. Sufficient conditions for exiting facet F_0

$$
\begin{array}{lcl}
(1) & & VV_s(v_0) \cap VV_{co}(v_0) \neq \emptyset,\ v_0 \in X = S(\mathbb{R}^n), \\
 & \Leftrightarrow & (\forall\ F_{n-1,j} \in F_{n-1}(X),\ h_j^T[Av_0 + Bu_0 + a - v_0] \leq 0); \\
(2) & & (\forall\ v_i \in V(F_0),\ VV_s(v_i) \cap VV_{co}(v_i) \neq \emptyset), \\
 & \Leftrightarrow & \left\{ \begin{array}{l} h_j^T[Av_i + Bu_i + a - v_i] \leq 0, \\ h_0^T[Av_i + Bu_i + a - v_i] > 0, \end{array} \right. ; \\
(3) & & \text{no steady state in } X, \\
 & \Leftarrow & [Av_0 + Bu_0 + a - v_0] \neq 0,\ \ \text{related to (1).}
\end{array}
$$

Control ACSP

Proposition. Time to exit

Upper bound on travel time from initial state to exit is,

$$
\begin{aligned}
& F_{n-1,1} \in F_{n-1}(X), \text{ exit facet, normal vector } h_1 \in \mathbb{R}^n, \\
& t_1 \leq [\beta - \alpha]/c_1, \\
\beta &= \max_{j \in \mathbb{Z}_{n_v}} h_1^T v_j, \quad \alpha = \min_{j \in \mathbb{Z}_{n_v}} h_1^T v_j, \\
c_1 &= \min_{j \in \mathbb{Z}_{n_v}} h_1^T [A v_j + B u_j + a - v_j].
\end{aligned}
$$

(automatica-v40p21-35, p. 29).

Input space dimension

If $m = n$ and $\mathrm{rank}(B) = n$
then the controllability conditions for the vectors are always satisfied.

Outline

Control ACSP

Problem. Control-to-facet
Consider affine control system on a polytope,

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\ & X \in P(\mathbb{R}^n),\ \text{full dimensional polytope}, \\ & U \in P(\mathbb{R}^m),\ \text{closed full-dimensional polyhedral set}, \\ & F_1 \in F_{n-1}(X),\ \text{a facet}. \end{aligned}$$

For all $x_0 \in X$ determine
a terminal time $t_1 \in [t_0, \infty)$ and an input $u : [t_0, t_1] \to U$
such that the state trajectory reaches the facet F_1 and exits the polytope:

1. $\forall t \in [t_0, t_1]$, $x(t) \in X$;
2. $x(t_1) \in F_1$, $t_1 \in T$, t_1 is smallest such time;
3. $h_1^T \frac{dx(t)}{dt}|_{t=t_1} > 0$, normal vector of F_1 points outward.

Preferably by continuous control law $g : X \to U$.

Control ACSP

Remarks

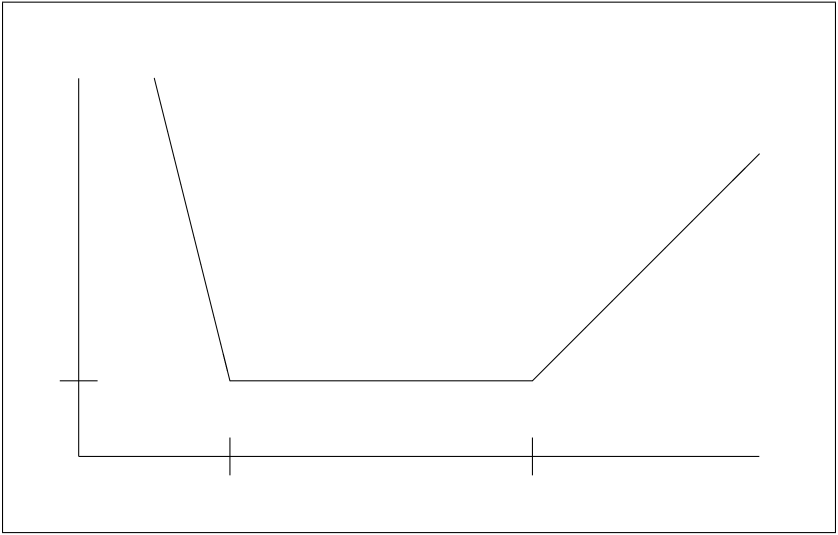
- Approach to control to facet
  not directly extendable from a simplex to a polytope.
- Reason, even if at all vertices there are constructed input vectors
  then there is no unique affine control law
  which achieves these input values.

$$\exists \, \{u_i \in U(v_i), \, \forall \, i \in \mathbb{Z}_k\}, \;\; [(Av_i + Bu_i + a) - v_i] \in VV_{co}(v_i),$$
$$\nexists \, g \in G_{aff} \;\text{ such that } \forall \, i \in \mathbb{Z}_k, \; u_i = g(v_i).$$

- Approach, triangulate polytope into simplices
  and determine an affine control law per simplex.
  Control law then called piecewise affine. Notation of class, G_{pwa}.
- Disadvantage of approach, many simplices in a triangulation.

Control ACSP

Figure. Continuous piecewise-affine function



Control ACSP

Def. Triangulation of a polytope

Define a triangulation of a polytope $P \subset P(\mathbb{R}^n)$,

$$
\begin{aligned}
\mathrm{Tr}(P) \quad &= \quad \{P_i \in P(\mathbb{R}^n),\ i \in \mathbb{Z}_k\}, \text{ such that,} \\
(1) \quad & \quad P = \cup_{i \in \mathbb{Z}_k} P_i, \\
(2) \quad & \quad \forall\ i \in \mathbb{Z}_k,\ \ P_i \text{ is a simplex,} \\
(3) \quad & \quad V(P) = \cup_{i \in \mathbb{Z}_k} V(P_i), \\
(4) \quad & \quad (\forall\ k \in \mathbb{Z}_{n_f},\ \forall\ F \in F_k(P)),\ F \in \mathrm{Tr}(P), \\
(5) \quad & \quad \forall\ i, j \in \mathbb{Z}_k,\ i \neq j, \\
& \quad \text{either } P_i \cap P_j \in (F_{k_1}(P_i) \cap F_{k_2}(P_j)) \text{ or } P_i \cap P_j = \emptyset.
\end{aligned}
$$

Remark. A triangulation may be called
a simplicial partition of the polytope with side conditions.

Control ACSP

Theorem. Sufficient condition for ACSP

Assume there exist $u_1, \ldots, u_M \in U$ satisfying (1.a - 2.b) and,

$$(2.c)\ \ \forall j \in \mathbb{Z}_M \backslash V(F_1):\ \ h_1^T(Av_j + Bu_j + a - v_j) > 0.$$

(a) There exists a continuous and piecewise affine map

$$\xi : X \to \mathbb{S}_+^M,\ x = \sum_{j=1}^{M} \xi(x)_j v_j \in X;$$

(b) Define

$$\psi : \mathbb{S}_+^M \to U,\ \psi(\lambda) = \sum_{j=1}^{M} \lambda_j u_j,$$
$$g : X \to U,\ g = \psi \circ \xi.$$

Then g is a continuous piecewise-affine control law and a solution to Problem ‘Control-to-facet’ for exit facet F_1.

Control ACSP

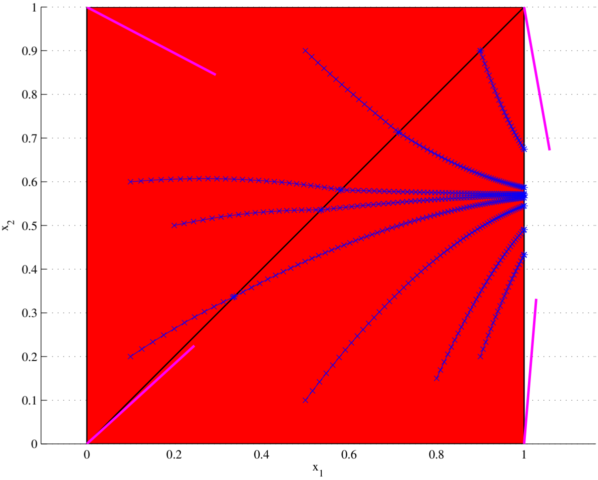
Procedure. Construction control law

1. Triangulate the polytope X into simplices as defined before. Algorithms [C.W. Lee, 1997].
2. For each simplex, define an affine control law.
3. The overall control law $g : X \to U$ is piecewise-affine and is continuous on common faces of simplices.

Remarks

1. Case multi-dimensional rectangles. Often occurs in applications. Necessary and sufficient differ only in terms of $<$ and $\leq$.
2. For online computation only the simplices are needed through which the state trajectory travels.

Control ACSP



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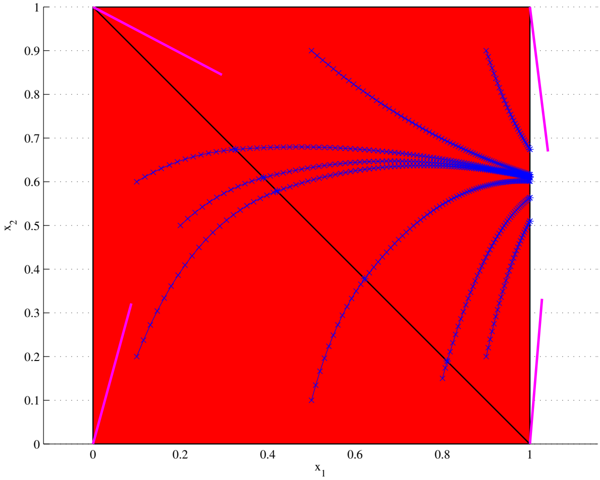

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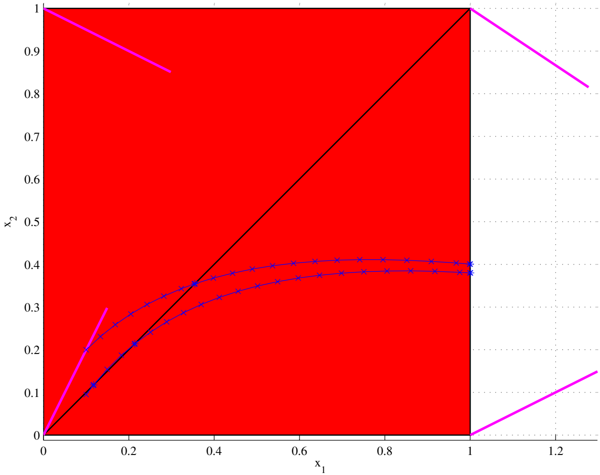


Control ACSP

Example. Unit cube in dimension four

Consider an affine control system on a polytope,

$$
\begin{aligned}
dx(t)/dt &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix} u(t) + \begin{pmatrix} 8 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \\
P &= \text{unit cube in } \mathbb{R}^4, \\
& \quad \text{exit facet is on hypersurface with } x_1 = 1, \\
& \quad \text{triangulation of unit cube with 56 simplices,} \\
g_8(x) &= F_8 x + r_8 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 3/2 & 1/2 & -1/2 \end{pmatrix} x + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Continuous piecewise-affine continuous control law.

Outline

Control PAHCSS

Problem. Control synthesis for PAHCSS 1

$$
\begin{aligned}
& && (Q, E, f, Q_0),\ \forall\, q \in Q, \\
dx_q(t)/dt &= && A(q)x_q(t) + B(q)u(t) + a(q),\ x_q(t_0) = x_{q,0}, \\
& && Q \text{ finite set},\ U \subset \mathbb{R}^m \text{ polytope}, \\
& && X_q \subset \mathbb{R}^{n_q},\ \text{simplex}, \\
& && G_q(e) \subset \partial X_q \text{ guards equal to or contained in facets}, \\
& && Q_u \subset Q \text{ unsafe locations},\ \ Q_s \subset Q \backslash Q_u \text{ start locations}, \\
& && q_d \in Q \backslash Q_u \text{ destination location (target location)}.
\end{aligned}
$$

Determine a piecewise-affine control law,

$$
\begin{aligned}
g_q(x) &= && F_q x + r_q,\ g_q : X_q \to U,\ \forall q \in Q, \\
& && \text{such that } \exists\ t_1 \in [t_0, \infty) \text{ and} \\
& && (t_0, q_s, x_{q_s,s}) \in T \times Q_s \times X_{q_s}\ \mapsto\ (t_1, q_d, x_{q_d,d}) \in T \times Q \times X_{q_d} \\
& && \text{either stay at target location or converge to fixed point } x_{q_d,d} \in X_{q_d}.
\end{aligned}
$$

Control PAHCSS

Problem. Control synthesis for PAHCSS 2

$$\forall\ q \in Q,\ \exists\ I(q) \subset \mathbb{Z}_+,\ \{F_{n-1,i} \in F_{n-1}(X),\ i \in I(q)\},$$

is called the set of admissable exit facets at discrete state $q \in Q$.

Control PAHCSS

Remark

Reachability of a state of the PAHCSS
is in general an undecidable problem.
Hence a sufficient condition for reachability
will be formulated which is decidable.

Approaches

1. Determine a control law such that
   the state trajectory of the closed-loop system
   leaves the simplex in finite time
   through only one or more of the prespecified facets.
2. For a destination state $x_d \in X_{q_d}$ determine a control law such that,

$$\lim_{t \to \infty} x_{q_d}(t; (t_2, x_{q_d,2})) = x_d,$$

   and never leaves the target location.

Control PAHCSS

Def. Types of control laws

(a) Affine control law G_a, often on a simplex $X \in P(\mathbb{R}^n)$.
$g(x) = Fx + r$ and $g : X \to U$.

(b) Discontinuous piecewise-affine control law G_{dcpwa}, figure.

$$g(x)|_{X_i} \quad = \quad F_i x + r_i, \;\; X = \cup X_i, \text{ triangulation.}$$

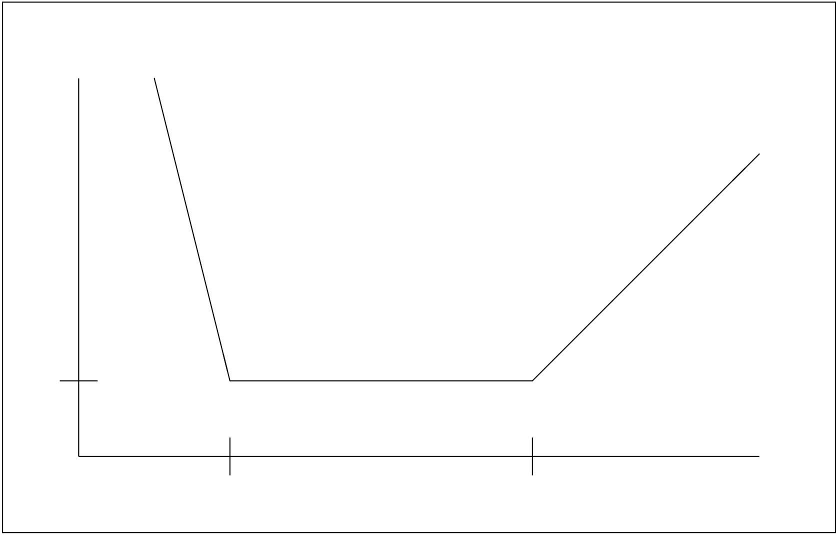
(c) Continuous piecewise-affine control law G_{cpwa}, figure.

$$\begin{aligned} g(x)|_{X_i} &= F_i x + r_i, \;\; X = \cup X_i, \text{ triangulation of full-dim. } X, \\ g(x)|_{X_i} &= g(x)_{X_j}, \text{ if } x \in X_i \cap X_j \in F_k(X_i) \cap F_k(X_j). \end{aligned}$$

(d) Continuous control law, G_c. $g : X \to U$.

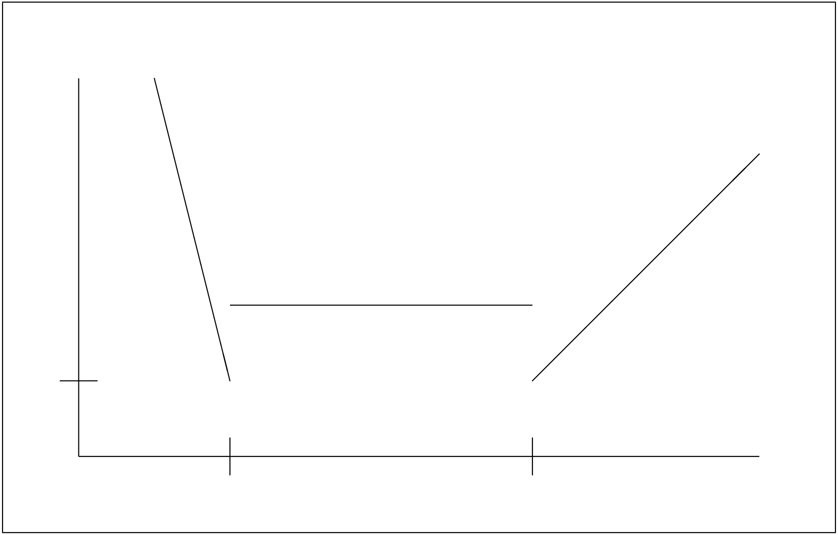
Control ACSP

Figure. Continuous piecewise-affine function



Control ACSP

Figure. Discontinuous piecewise-affine function



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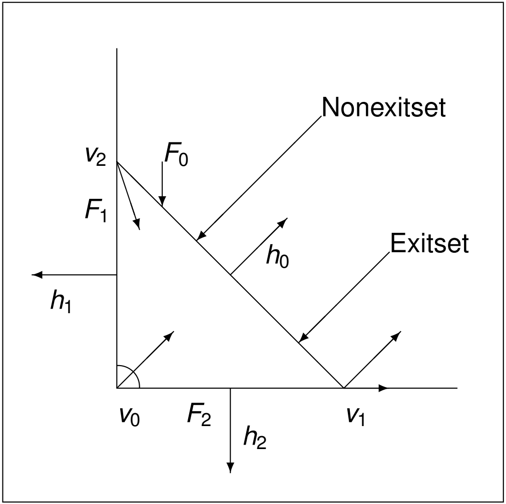


Control ACSS

Remarks

(1) Attempts to exit.

$$\begin{aligned} & X \in P(\mathbb{R}^n),\ F_1 = \text{convh}(\{v_j, j \in V(F_1)\}, \\ & h_1^T(Ax + a - x) > 0,\ \forall\, x \in X, \\ \Leftrightarrow \quad & h_1^T(Av_j + a - v_j) > 0,\ \forall\, v_j \in V(F_1). \end{aligned}$$

Latter regarded as too strong a condition, see figure.
(2) Could consider condition,

$$h_1^T(Av_1 + a - v_1) > 0,\ \ h_0^T(Av_j + a - v_j) \leq 0,\ \forall\, v_j \in V(F_1)\backslash\{1\}.$$

(3) Existence of state trajectory outside polytope on an interval.
(K.K. Lee, A. Arapostathis (1987), Def. 6, scl-v9p89-96).

$$\exists\, \epsilon \in (0, \infty),\ \forall\, t \in (t_1, t_1 + \epsilon),\ \ x(t) \notin X.$$

Solution of differential equation defined in a neighborhood of X (outside X).

Control PAHCSS

Def. Exit set

$$\begin{aligned} dx(t)/dt &= Ax(t) + a,\ x(0) = x_0 \in X, \\ & F_{n-1,i} \in F_{n-1}(X), \\ \mathrm{ExitSet}(F_{n-1,i}) &= \mathrm{cl}\{x \in F_{n-1,i} | h^T_{F_{n-1,i}}[Ax + a - x] > 0\}. \end{aligned}$$

Facet $F_{n-1,i}$ called blocked if $\mathrm{ExitSet}(F_{n-1,i}) = \emptyset$.

Def. Leaving state set

State trajectory is said to leave the state set at $t_1 \in [0, \infty)$ if

$$\begin{aligned} &(1) \quad \forall\, t \in [0, t_1),\ x(t; (0, x_0)) \in X, \\ &(2) \quad \exists\, \epsilon \in (0, \infty),\ \forall\, t \in (t_1, t_1 + \epsilon),\ x(t) \notin X. \end{aligned}$$

State trajectory crosses facet $F_{n-1,i} \in F_{n-1}(X)$ at time $t_1 \in [0, \infty)$ if, in addition, $x(t_1; (0, x_0)) \in \mathrm{ExitSet}(F_{n-1,i})$.

Control PAHCSS

Lemma.
If the state trajectory leaves the simplex X at $\hat{x} \in \partial(X)$ then $\exists\ F_{n-1,i} \in F_{n-1}(X)$ such that $\hat{x} \in \mathrm{ExitSet}(F_{n-1,i})$.

Proposition.
If the state trajectory leaves leaves the simplex at,

$$x(t_1) = x_1 \in F_{n-1,i} \cap F_{n-1,j},\ i \neq j,$$

then the state trajectory belongs to the exit set of an admissable facet. (ieeetac-v51p938-948, Property 4.8)

Control PAHCSS

Problem. Control-to-a-facet

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\
& X \subset \mathbb{R}^n \text{ simplex},\ U \subset \mathbb{R}^m \text{ polytope}, \\
& \exists\, q \in Q,\ I(q) \subseteq \mathbb{Z}_N = \{0,\ 1,\ \ldots,\ N\}, \\
& \{F_i \in F_{n-1}(X),\ \forall\, i \in I(q)\} \text{ admissable exit facets.}
\end{aligned}
$$

Determine an affine control law,

$$
\begin{aligned}
g(x) &= Fx + r, \\
dx(t)/dt &= (A + BF)x(t) + (a + Br),\ x(t_0) = x_0,
\end{aligned}
$$

closed-loop system, such that,
if the state trajectory of the closed-loop system leaves the simplex X
then it does so through one of the admissable exit facets,
$F_{n-1,i} \in F_{n-1}(X)$ for $i \in I(q)$.

PAHCSS

Theorem. Control-to-a-facet
There exists an affine control law for this problem
if and only if

$$\begin{aligned}
& \exists\ u_0, \ldots, u_n \in U \ \text{ such that}, \\
& h_i^T (Av_j + Bu_j + a - v_j) \leq 0, \ \forall i \in \mathbb{Z}_n \backslash I(q), \ \forall j \in \mathbb{Z}_n \backslash \{i\}. \\
& \text{Let } (F, r) \text{ be the unique solution of the equation}, \\
& \begin{pmatrix} v_0^T & 1 \\ \vdots & \vdots \\ v_n^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ r^T \end{pmatrix} = \begin{pmatrix} u_0^T \\ \vdots \\ u_n^T \end{pmatrix}, \\
g(x) \ = \ & Fx + r, \ \text{is affine control law and solution to problem.}
\end{aligned}$$

(ieeetac-v51-938-948, Th. 4.12)

Remarks
Linear inequalities solvable by a computation.

PAHCSS

Problem. Control-to-exit
Consider,

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\ & X \subset \mathbb{R}^n \text{ simplex, } I(q) \subset \mathbb{N}_n, \\ & \{F_i \in F_{n-1}(X),\ \forall\, i \in I(q)\} \text{ admissable exit facets.} \end{aligned}$$

Determine an affine control law,

$$g(x) = Fx + r,$$

such that the state trajectory of the closed-loop system
leaves the polytope X via an admissable exit facet in finite time.

PAHCSS

Theorem. Control-to-exit

$$\begin{aligned} U_j &= \left\{ u \in U | h_k^T (Av_j + Bu + a - v_j) \leq 0, \ \forall \ k \in \mathbb{Z}_N \backslash (I(q) \cup \{j\}) \right\}, \\ V(U_j) & \quad \text{vertices of } U_j, \ \ \forall j \in \mathbb{Z}_N. \end{aligned}$$

The problem is solvable if and only if

$$\begin{aligned} & \forall j \in \mathbb{N}_N, \ \exists w_j \in V(U_j), \text{ such that,} \\ 0 \ \ \notin & \ \ \mathrm{convh}(\{Av_j + Bw_j + a | \forall \ j \in \mathbb{N}_N\}). \end{aligned}$$

(ieeetac-v51p938-948, Th. 4.17)

Remarks

Linear equalities to be checked for existence of a solution.

PAHCSS

Theorem. Existence of steady states

Consider an autonomous affine system on a polytope,

$$dx(t)/dt = Ax(t) + a, \; x(t_0) = x_0, \; X \subset \mathbb{R}^n.$$

There exists a steady state (fixed point),

$$0 = Ax_s + a, \quad x_s \in X,$$

if and only if

$$\exists \; x_0 \in X \text{ such that } \forall t \in [t_0, \infty), \; x(t; (t_0, x_0)) \in X.$$

(ieeetac-v51p938-948, Th. 3.1)

Proof. $\Rightarrow$ By $x_0 = x_s$.
$\Leftarrow$ Apply Brouwer's fixed point theorem.
Remark. Distinguish a steady state $x_s \in X$ from an equilibrium state $x_e = 0$.

PAHCSS

Problem. Stabilization at a steady state

Consider the affine control system on a simplex,

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\ & X \subset \mathbb{R}^n \text{ simplex, } U \subset \mathbb{R}^m \text{ polytope, } x_s \in X \text{ steady state.} \end{aligned}$$

Determine an affine control law,

$$g(x) = Fx + r,$$

such that,

1. control law g is admissable: $\forall x \in X$, $g(x) \in U$;
2. state trajectory is admissable: $\forall t \in T$, $x(t; (t_0, x_0)) \in X$;
3. state trajectory converges to steady state: $\lim_{t \to \infty} x(t; (t_0, x_0)) = x_s$.

PAHCSS

Theorem. Stabilization-to-a-steady-state

Problem is solvable if and only if,

$$\exists\ u_0, \ldots, u_N \in U, \text{ such that}$$

$$(1) \qquad h_i^T(Av_j + Bu_j + a - v_j) \leq 0,\ \forall j \in \mathbb{N}_n,\ \forall i \in \mathbb{N}_n \backslash \{j\};$$

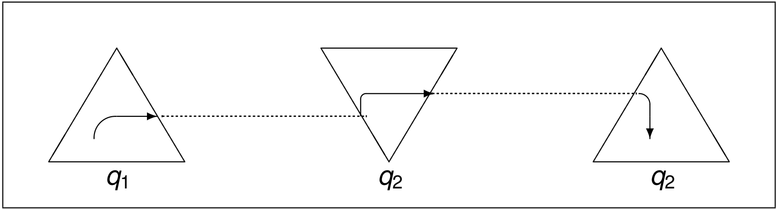
$$(2) \qquad B\sum_{j=0}^{n} \mu_j u_j = -Ax_s - a;\ x_s = \sum_{i=0}^{n} \mu_i v_i,$$

$$(3) \qquad \mathrm{span}(\{Av_j + Bu_j + a - v_j | \forall j \in \mathbb{N}_n\}) = \mathbb{R}^n.$$

Notation $x_s = \sum_{j=0}^{n} \mu_j v_j$.
(ieeetac-v51p938-948, Th. 4.19)

Control PAHCSP

Figure of PAHSP



PAHCSS

Def. Discrete-event system

$$
\begin{aligned}
DES &= (Q, E, f),\ Q \text{ state set, } E \text{ event set,} \\
& \quad f_d : \mathrm{Dom}(f_d) \subset (Q \times E) \to Q \text{ transition function,} \\
E_q &= \{e \in E | (q, e) \in \mathrm{Dom}(f_d)\}, \\
& \quad (q_0, q_1, \ldots, q_n),\ q_i = f(q_{i-1}, e_i). \\
& \quad \text{For control law } k_q \text{ define,} \\
(s_q, m_q) & \quad \text{local supervisor},\ s_q \in E_q,\ m_q \in \{\bot, \top\},\ \forall q \in Q, \\
& \quad e \in s_q \text{ if } x_q(t) \text{ leaves } X_q \text{ through guard/facet } G_q(e); \\
m_q &= \top,\ \text{if there exists a steady state } x_{q,s} \in X_q.
\end{aligned}
$$

PAHCSS

Problem. Reach-avoid problem for a PAHCSS

$$\begin{aligned} DES &= (Q, E, f), \;\; Q_s \subset Q \text{ starting states,} \\ & \quad q_d \in Q \text{ destination state, } \;\; Q_u \in Q \text{ unsafe states.} \end{aligned}$$

Determine a supervisor S such that,

1. S/DES is nonblocking;
2. $q_0 \in Q_s$ and there exists an integer $n \in \mathbb{Z}_+$ such that $q_n = q_d$;
3. $\forall\ i \in \mathbb{N}_n$, $q_i \notin Q_u$.
4. Reach-avoid-stabilize, in addition, $s_{q_d} = \emptyset$ and $m_{q_d} = \top$.
5. Reach-avoid-converge, in addition, $\lim_{t\to\infty} x_{q_d}(t) = x_s$.

PAHSS

Algorithm. Reach-avoid problem

Compute the co-reachable set,
(like dynamic programming with backward recursion).

1. If $q_d \in Q_u$ then terminate.
2. $Q_0 = \{q_d\}$ and choose a set of supervisors S_{q_d} and $i = 0$.
3. While not $(Q_s \subset Q_j$ or $Q_j = Q_{j-1})$ do $i = i + 1$,

$$
\begin{array}{ll}
(3.1) & Q_i = Q_{i-1} \cup \\
& \cup \left\{ \begin{array}{l} q \in Q \backslash Q_u | \exists (s_q, m_q) \in S_q \\ \text{such that } f(q, s_q) \subset Q_{i-1} \end{array} \right\}; \\
(3.2) & S_q \text{ local supervisor found in (3.1)}
\end{array}
$$

Output: Q_j and $\{s_q \in S_q, q \in Q_j\}$.

Theorem. Reach-avoid problem

If Algorithm 6.2 terminates with $Q_s \subset Q_j$
then there exists a solution to Problem Reach-avoid problem for PAHSS.
(ieeetac-v51p938-948, Th. 5.3)

PAHCSS

Example. Reach-avoid-stabilize

$$
\begin{aligned}
dx_q(t)/dt &= A(q)x_q(t) + B(q)u(t) + a(q),\ x_q(t_0) = x_0, \\
Q &= \{q_1, \ldots, q_5\},\ E = \{e_1, \ldots, e_5\}, \\
X_q &= \left\{x \in \mathbb{R}^2 | x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\right\},\ \forall q \in Q, \\
G_{q_1}(e_2) &= F_3,\ G_{q_1}(e_3) = F_2,\ G_{q_1}(e_4) = F_1,\ \text{etc.} \\
\dot{x}_{q_1} &= \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} x_{q_1}(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \text{etc.} \\
Q_s &= \{q_1\},\ q_d = q_5,\ Q_u = \{q_4\}.
\end{aligned}
$$

(Continued on next slide)

PAHCSS

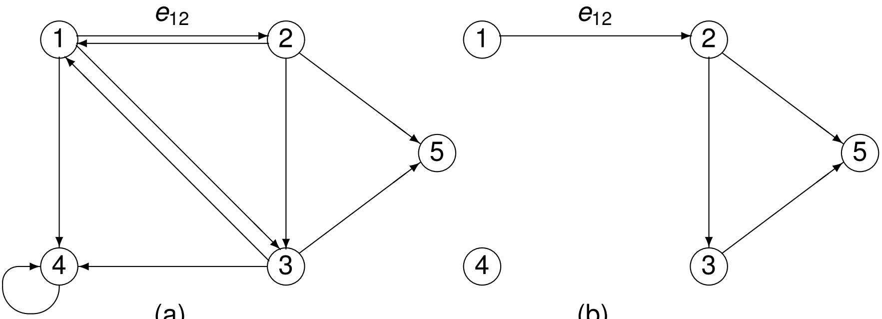
Example. (Continued)

Control law specified by:

$$
\begin{aligned}
g_{q_1}(x) &= 1, \quad q_1 \mapsto q_2, \\
g_{q_2}(x) &= 0, \quad q_2 \mapsto q_3, q_5, \\
g_{q_3}(x) &= 1, \quad q_3 \mapsto q_5, \\
g_{q_5}(x) &= -x_1 - \frac{3}{4}x_2 + \frac{1}{2}, \quad q_5 \mapsto q_5.
\end{aligned}
$$

PAHCSS

Diagram of discrete-event system
(a) Open-loop system; (b) Closed-loop system.

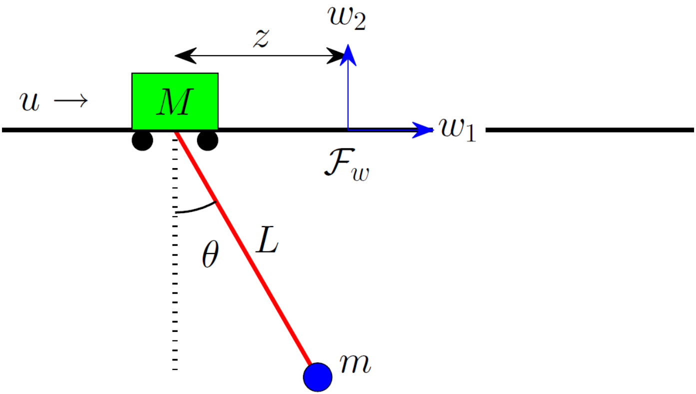


(a) (b)

Motivation

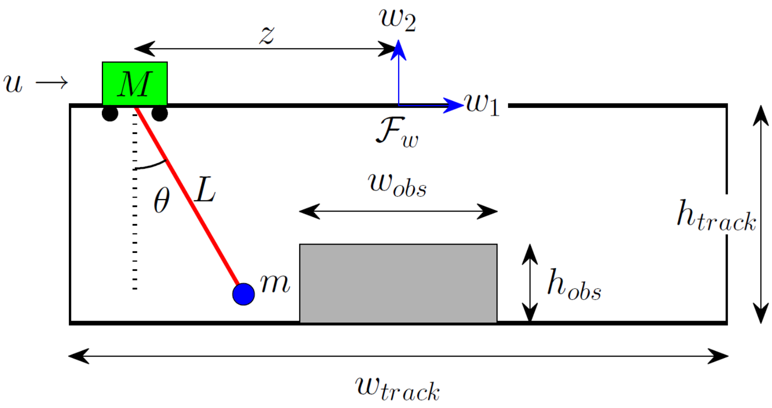
Example. PAHSP – Figure

Source: cdc2014-p3609-3614 (M. Vukoslavljev, M.E. Broucke).



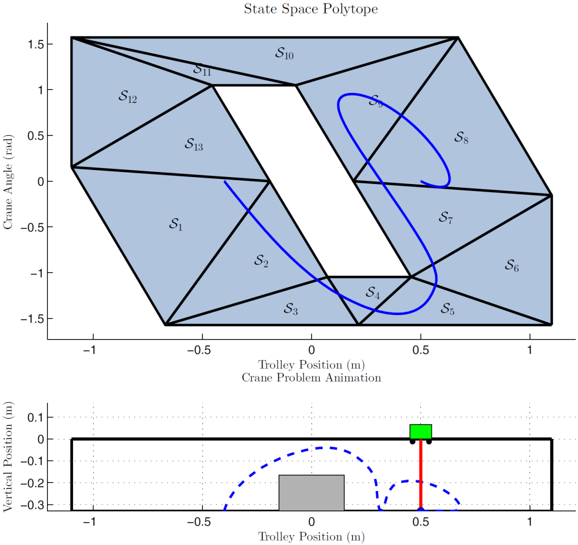
Motivation

Example. PAHSP – Figure



Motivation

Example. PAHSP – Figure



Outline

Extensions

Reach Control Contributions

Research Group of M.E. Broucke (U. of Toronto) with many students period 2005–now. Main contributions to control theory:

1. Characterization of subsets of potential steady states of an affine control system on a simplex. (siamjco-v48p3482-3500).
2. Control synthesis for reach control problem of an affine control system on a simplex. (siamjco-v52p3261-3286).
3. Sufficient conditions for solution of reach control problem for an affine control system on a polytope. (automatica-v55p108-115).

Extensions

Contributions of Calin Belta etc.

- Cooperation with Luc Habets (TUE).
- Control to facet of multi-affine systems on rectangles.
- Applications to engineering and to biological systems.