Control of Piecewise-Affine Hybrid Systems – Lecture 3 Abstraction and Control Synthesis in Case of Partial Observations and Generalizations

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Outline

Outline

Abstraction

Problem. Abstraction

Abstract the behavior of a piecewise-affine system on a polytope
by the behavior of an automaton
such that the behavior of the automaton
explains part of the behavior of the system.

Motivation

- Biochemical reaction systems of state-space dim. 10 – 100 or more.
  Power systems of state-space dim. 40 – 300.
- Analysis of the behavior of such a system is difficult.
  For example, sets of eigenvalues of the Jacobian at steady states.
  Computations take much time and
  simulations are only an approximation.
- Abstracted behavior by automaton can be computed.
  For example, whether a state variable exceeds a threshold.
- Examples of biochemical reaction systems indicate usefulness.

Abstraction

Procedure. Abstraction and its use

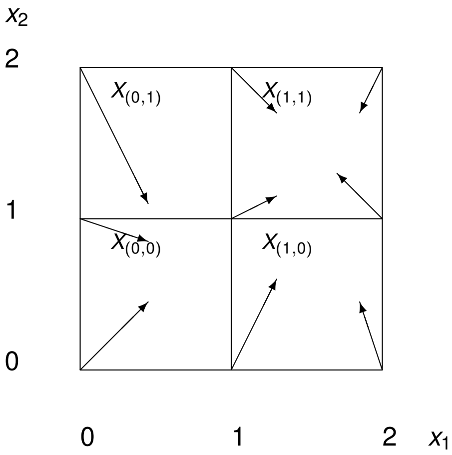
1. Abstract the behavior of the system by the behavior of the automaton.
2. Analyse the behavior of the automaton.
3. Relate the behavior of the automaton to that of the system.

Remarks. History

Investigators: H. de Jong (INRIA Alpes) and
J.L. Gouzé (INRIA Sophia Antipolis).
C. Belta etal (Boston University).
D. Safranek etal (Masaryk University Brno, CZ),
CWI Research Group Control and System Theory.

Abstraction – Figure

Example. Affine system in dimension 2 – Figure



Abstraction

Example. Affine system in dimension 2

$$
\begin{aligned}
dx(t)/dt &= \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 6.8 \\ 6.5 \end{pmatrix},\; x(0) = x_0, \\
X &= X_{(0,0)} \cup X_{(1,0)} \cup X_{(0,1)} \cup X_{(1,1)}, \\
& \quad X_{(0,0)} = [0,1]^2,\; X_{(0,1)} = [0,1] \times [1,2], \text{etc.} \\
Q &= \{q(0,0), q(0,1), q(1,0), q(1,1)\}, \\
E &= \{e(0,0,1,0),\; ...\}, \\
& \quad q(i,j) \stackrel{e(i,j,k,l)}{\rightarrow} q(k,l), \\
q(k,l) &= f(q(i,j), e(i,j,k,l)),\; \forall\, (i,j),\; (k,l).
\end{aligned}
$$

Abstraction

Def. Automaton

$$A \quad = \quad (Q, E, f, Q_0),$$

Q finite state set, E finite event set, $Q_0 \subseteq Q$ set of initial states,
$f : Q \times E \to Q$ transition function, a partial function.

Def. State trajectory of automaton

$$\begin{aligned}
&(q_0,\ q_1,\ q_2,\ \ldots,\ q_k) \in Q^*, \\
&\text{if } \exists\ s = e_1\ e_2\ \ldots\ e_k \in E^*, \\
&q_0 \in Q_0,\ q_{i+1} = f(q_i, e_i),\ \forall\ i = 0, \ldots, k-1.
\end{aligned}$$

May also be defined for strings of infinite length denoted by set Q^ω.

Abstraction

Def. Admissable partitioning of X

Consider a closed full-dimensional polytope $X \in P(\mathbb{R}^n)$.
Call a partitioning of X admissable if

$$\begin{aligned} X_{part}(X) &= \{X_i \subset X, \ \forall\, i \in \mathbb{Z}_m\}, \\ (1) & \quad X_i \text{ closed full-dim. polytope}, \\ (2) & \quad X = \cup_{i \in \mathbb{Z}_m} X_i, \\ (3) & \quad ((\forall\, i, j \in \mathbb{Z}_m,\ i \neq j), \text{ either } X_i \cap X_j \text{ empty or} \\ & \quad X_i \cap X_j \in F_k(X_i) \cap F_k(X_j)). \end{aligned}$$

Called admissable rectangular partition
if X is a rectangle and $((\forall\, i \in \mathbb{Z}_m),\ X_i$ are rectangles).

Abstraction

Def. Piecewise-affine system on a polytope

$$
\begin{aligned}
& (X, X_{part}(X), x_0, t_0, f_c), \\
& X \subset \mathbb{R}^n, \text{ full-dim. polytope}, \\
& X_{part}(X), \text{ admissable partition}, \\
& x_0 \in X, \; t_0 \in T = \mathbb{R}, \\
& f_c : X \to \mathbb{R}^n, \text{ continuous piecewise-affine map}, \\
dx(t)/dt &= f_c(x(t)), \; x(t_0) = x_0, \\
& f_c|_{X_i}(x) = A_i x + a_i.
\end{aligned}
$$

If $f_c|_{X_i}$ are all identical then no Filippov solution is needed.
Otherwise, use concept of Filippov solution. (A.F. Filippov)
Figure on blackboard.

Abstraction

Def. Abstraction of system by an automaton

Define the abstraction of a piecewise-affine system by an automaton with the relation that every solution of the system corresponds to a trajectory of the automaton.

$$
\begin{aligned}
|Q| &= |X_{part}(X)|, \\
& \quad \pi : X_{part}(X) \to Q, \text{ invertible map;} \\
& \quad x(t;(t_0, x_0)) \in X_i \;\Leftrightarrow\; q_i = \pi(X_i) \in Q; \\
& \quad x : [t_0, t_1] \to X \\
&\Rightarrow (q_0, q_1, \ldots, q_k, \ldots) \in Q^* \cup Q^{\omega}.
\end{aligned}
$$

Abstraction

Procedure. Abstraction

1. For any simplex/polytope $X_i \in X_{part}(X)$, determine the exit facets of X_i;
2. Determine whether for any initial state in $X_i \in X_{part}(X)$, the state trajectory will leave the simplex/polytope in finite time or stay in the simplex X_i forever.
3. For any simplex/polytope $X_{part}(X)$, and if the state trajectory exits from X_i, determine to which other polytopes $X_j \in X_{part}(X)$ it proceeds.
4. Define the transition of the automaton as,

$$\begin{aligned} & q_j = f_d(q_i, e(i,j)) \in Q, \\ & \text{if } \exists\ X_j \in X_{part}(X) \text{ such that } x(t_1^-) \in X_i,\ x(t_1^+) \in X_j, \end{aligned}$$

if from simplex X_i the state trajectory exits X_i and then proceeds to simplex X_j.

Abstraction

Def. Exit facets of X_i

Consider $X_i \in X_{part}(X)$.
Call a facet $F \in F_{n-1}(X)$ an exit facet
if there exists a state trajectory
of the piecewise-affine control system starting in X_i
that leaves the simplex X_i in finite time by crossing facet F.

$$\exists\ x_F \in F \in F_{n-1}(X_i),\ \text{such that}\ \ h_F^T(A_i x_F + a_i - x_F) > 0.$$

Abstraction

Theorem. State trajectory leaves simplex in finite time

$$
\begin{aligned}
dx(t)/dt &= A_i x(t) + a_i,\ x(t_0) = x_0, \\
& \quad X_i \subset X,\ \text{closed full-dimensional polytope}, \\
& \quad ((\exists\, x_0 \in X_i,\ \forall\, t \in [t_0, \infty)), x(t; (t_0, x_0)) \in X_i)), \\
\Leftrightarrow & \quad ((\exists\, x_s \in X_i),\ \text{such that}, A_i x_s + a_i = 0), \\
\Leftrightarrow & \quad 0 \in \mathrm{convh}(\{A_i v + a_i \in \mathbb{R}^n | \forall\, v \in V(X_i)\}).
\end{aligned}
$$

Condition solvable by checking existence of a solution
to a set of linear equations.

(ifacwc2011, PJC etal, Th. 8+9).

Abstraction

Procedure. Does the state trajectory leave the simplex?

Cases:

1. If for $X_i \in X_{part}(X)$, for $v_j \in V(X_i)$,
   $\forall\ F_k \in F_{n-1}(X_i)$, $h_k^T[A_i v_j + a_i - v_j] \leq 0$
   then the state trajectory remains in X_i forever.
2. If for $X_i \in X_{part}(X)$
   there does not exist a steady state $x_s \in X_i$ such that $A_i x_s + a_i = 0$
   then all trajectories starting in X_i leave X_i in finite time.
3. If for $X_i \in X_{part}(X)$
   there exists a steady state $x_s \in X_i$ such that $A_i x_s + a_i = 0$
   and there exists a $F \in F(X_i)$ and a $v_j \in V(F)$
   such that $h_F^T[A_i v_j + a_i - v_j] > 0$,
   then either
   (1) exists a $x_0 \in X_i$ such that for all $t \in T$, $x(t;(t_0, x_0)) \in X_i$ or
   (2) exists $x_0 \in X_i$ such that $x(t;(t_0, x_0))$ leaves X_i in finite time.

Abstraction

Procedure. Construction automaton
Details omitted.

Calculations for the abstraction simplify
if X and all elements of $X_{part}(X)$ are rectangles.

Abstraction

Def. Abstraction sufficiency and necessity

Consider a piecewise-affine hybrid system on a polytope and its abstraction,

$$(X, X_{part}(X), x_0, t_0, f_c), \;\; A = (Q, E, f_d, Q_0).$$

(a) Global sufficiency of the abstraction is said to hold if for all initial states $x_0 \in X_0$ the state trajectory $\{x(t; (t_0, x_0)), \; t \in T\}$ corresponds to a discrete trajectory of the automaton A as defined in the definition.

(b) Global necessity of the abstraction is said to hold if for any discrete state trajectory of the automaton there exists an initial state $x_0 \in X_0$ such that $\{x(t; (t_0, x_0)), \; t \in T\}$ corresponds to the discrete-state trajectory.

Abstraction

Theorem. Relation system and its abstraction

(a) The relation
between a piecewise-affine hybrid system and its abstraction
is globally sufficient.

(b) There exists an example of a piecewise-affine hybrid system
for which the abstraction is not globally necessary.

(ifacwc2017, PJC etal., Th. 14).

Remarks on abstraction
More experience needed.

Outline

Example. Cart with pendulum – Partial Observations

Observe the position of the cart and of the center of pendulum.
One does not observe their speeds, neither of car nor of angle.

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t),\ x(0) = x_0, \\ y(t) &= \begin{pmatrix} x_1(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} s(t) \\ s(t) + L_1\phi(t) \end{pmatrix} \\ &= Cx(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t). \end{aligned}$$

Control Partial Observations

Def. Linear control system with partial observations

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t),\ x(0) = x_0 \in X = \mathbb{R}^n, \\
y(t) &= Cx(t) + Du(t), \\
& \quad Y = \mathbb{R}^p,\ p \in \mathbb{Z}_+,\ y : T \to Y, \\
& \quad C \in \mathbb{R}^{p \times n},\ D \in \mathbb{R}^{p \times m}.
\end{aligned}
$$

y called the output function, C output matrix.

Remarks

That the full state cannot be observed
is true for most engineering control systems.
Hence control with partial observations is motivated.

Control Partial Observations

Def. Control law with partial observations

Define a static-output-feedback control law,

$$u(t) = g(y(t)), \;\; g : Y \to U.$$

Define a dynamic-output-feedback control law
with linear dynamics, by the control system,

$$\begin{aligned}
dz(t)/dt &= A_1 z(t) + B_1 u(t) + Ky(t), \; z(0) = z_0, \\
u(t) &= g(z(t), y(t)); \\
& \quad Z = \mathbb{R}^k, \; k \in \mathbb{Z}_+, \; K \in \mathbb{R}^{k \times p}, \text{ etc.} \\
& \quad g : Z \to U, \\
& \quad g \text{ is called a linear control law if} \\
g(z) &= Fz, \;\; u(t) = Fz(t).
\end{aligned}$$

Linear system with state z often called
an observer of the original system.

Control Partial Observations

Def. Closed-loop system

Consider ...
Consider a dynamic-output-feedback control law.
Define the associated closed-loop system by the equations,

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A & BF \\ KC & A_1 + B_1 F \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}.$$

Problem. Control with partial observations

Does there exist a static/dynamic-output-feedback control law
such that the closed-loop system satisfies the control objectives:
(1) stability, (2) performance optimized, and (3) robustness.

Control Partial Observations

Def. Controllability

A linear system is called controllable if

$$\begin{aligned} & \forall\ x_0,\ x_1 \in X = \mathbb{R}^n, \\ & \exists\ u : [t_0, t_1] \rightarrow U = \mathbb{R}^m,\ \text{such that,} \\ & x(t_0) = x_0 \ \stackrel{u}{\rightarrow}\ x(t_1) = x_1, \\ dx(t)/dt \ &= \ Ax(t) + Bu(t),\ \ x(t_0) = x_0,\ \ x(t_1) = x_1. \end{aligned}$$

Proposition. Characterization controllability

The linear control system is controllable if and only if

$$n \ = \ \text{rank}\left(\begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix} \right),\ \ \text{where, } X = \mathbb{R}^n.$$

Control Partial Observations

Def. Observability

A linear system is called observable if

$$\begin{aligned}
& & \exists\ [0,t_1],\ \forall\ x_0 \in X, \\
& & x_0 \mapsto \{y(t) \in Y,\ \forall\ t \in [0,t_1]\} \text{ is injective;} \\
& \Leftrightarrow & x_0 \mapsto \{\frac{d^k y(t)}{dt^k}|_{t=0},\ \forall\ k \in \mathbb{N}_{n-1}\} \text{ is injective;} \\
dx(t)/dt & = & Ax(t),\ \ x(t_0) = x_0, \\
y(t) & = & Cx(t).
\end{aligned}$$

Proposition. Characterization controllability

The linear control system is observable if and only if

$$n = \operatorname{rank}\left(\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}\right), \text{ where, } X = \mathbb{R}^n.$$

Control Partial Observations

Theorem. Existence dynamic control law

For any arbitrary selection of eigenvalues of the closed-loop system
there exists a dynamic-output-feedback control law
transfering the system from an initial state to a terminal state
if and only if
(1) the linear system is controllable and
(2) the linear system is observable.

Control Partial Observations

Procedure. Control synthesis

1. Check if the system is controllable.
   If so, then construct a matrix $F \in \mathbb{R}^{m \times n}$ such that,

$$\mathrm{spec}(A + BF) \subset \mathbb{C}^- = \{c \in \mathbb{C} | \Re(c) < 0\}.$$

2. Check if the system is observable.
   If so, then construct a matrix $K \in \mathbb{R}^{n \times p}$ such that,

$$\mathrm{spec}(A + KC) \subset \mathbb{C}^-.$$

3. Define the dynamic-output-feedback control law,

$$\begin{aligned} dz(t)/dt &= Az(t) + Bu(t) + K[y(t) - Cz(t)],\ z(0) = z_0, \\ u(t) &= Fz(t), \\ dz(t)/dt &= (A + BF - KC)z(t) + Ky(t),\ z(0) = z_0, \\ g(z) &= Fz. \end{aligned}$$

Control Partial Observations

Theorem. Closed-loop system

Consider the closed-loop system associated with
the linear system and the synthesized dynamic control law. Then,

$$\begin{aligned}
\begin{pmatrix} dx(t)/dt \\ de(t)/dt \end{pmatrix} &= \begin{pmatrix} A+BF & BF \\ 0 & A+KC \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \\
&= A_{x,e} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix}, \ \begin{pmatrix} x(0) \\ e(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ e_0 \end{pmatrix}, \\
u(t) &= Fz(t) = F(x(t)-e(t)), \ \ e(t) = x(t) - z(t); \\
\text{spec}(A_{x,e}) &= \text{spec}(A+BF) \cup \text{spec}(A+KC) \subset \mathbb{C}^-.
\end{aligned}$$

Eigenvalues of $A_{x,e}$ can be chosen by control designer.

Outline

CPAHCSP.PO

Def. Affine control system with output

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a = f(x(t)),\ x(0) = x_0, \\
y(t) &= Cx(t), \\
& X \in P(\mathbb{R}^n),\ \text{state polytope}, \\
& U \in P(\mathbb{R}^m),\ \text{input polytope}, \\
Y &= CX = \{Cx \in \mathbb{R}^p,\ \forall\ x \in X\},\ \ \text{output polytope}.
\end{aligned}
$$

CPAHCSP.PO

Def. Exit set

$$
\begin{aligned}
& F_i = F_{n-1,i} \in F_{n-1}(X), \\
\mathrm{ExitSet}(F_i) &= \text{cl}\{x \in F_i | h_i^T[f(x) - x] > 0\}, \text{ exit set; if} \\
F_i &= \{x \in F_i | h_i^T[f(x) - x] \leq 0\} \text{ then,} \\
\mathrm{ExitSet}(F) &= \emptyset, \text{ facet is called blocked.}
\end{aligned}
$$

CPAHCSP.PO

Def. Trajectory leaves

Trajectory of control system leaves polytope P
by crossing facet $F \in F_{n-1}(X)$ at time $t_1 \in T = [0, \infty)$ if

1. $\forall\ t \in [0, t_1],\ x(t; (0, x_0)) \in X$;
2. $x(t_1; (0, x_0)) \in \mathrm{ExitSet}(F)$;
3. $\exists\ \epsilon \in (0, \infty), \forall\ t \in (t_1, t_1 + \epsilon),\ x(t_1; (0, x_0)) \notin X$.

(2) requires $x(t_1; (0, x_0)) \in F$.
(3) needs definition of $x(.; (0, x_0))$ on neighborhood of X;
possibly due to (K.K. Lee and A. Arapostathis (1987), scl-v9p89-96).

CPAHCSP.PO

Def. Output feedback

Static-output-feedback control law
$g : Y \to U$, Lipschitz continuous, $u(t) = g(y(t))$.

Affine static-output-feedback control law
if $g(y) = Fy + r$, $F \in \mathbb{R}^{m \times p}$, and $r \in \mathbb{R}^m$.

Piecewise-affine static-output-feedback control law if

$$\begin{aligned} & \exists \, \{Y_i \subseteq P(Y), \, \forall \, i \in I\}, \text{ polyhedral subdivision of } Y, \\ & \exists \, \{(F_i, r_i) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m, \, \forall \, i \in I\} \text{ such that,} \\ g(y)|_{Y_i} \quad = \quad & F_i y + r_i. \end{aligned}$$

Could be continuous at facets of the polyhedral subdivision of Y.
Admissable control law if $g(Y) \subseteq U$.

CPAHCSP.PO

Def. Closed-loop system

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bg(y(t)) + a \\
&= Ax(t) + Bg(Cx(t)) + a, \; x(0) = x_0, \\
y(t) &= Cx(t).
\end{aligned}
$$

If g is piecewise affine
then the closed-loop system is a piecewise-affine system.

CPAHCSP.PO

Problem. Blocking facets

Consider $F_{bl} \subseteq F_{n-1}(X)$, the subset of blocked facets.
Determine a piecewise-affine static-output-feedback control law such that the state trajectory will not exit from any $F \in F_{bl}$.

Problem. Control-to-exit

Consider $F_{exit} \subseteq F_{n-1}(X)$, the subset of potential exit facets.
Determine a piecewise-affine control law such that the closed-loop system satisfies,

$$\begin{aligned} &\forall\, x_0 \in X,\ \exists\, F \in F_{exit},\ \exists\, t_1 \in [0, \infty), \\ &\text{state trajectory leaves } X \text{ with } \ x(t_1; (0, x_0)) \in \mathrm{ExitSet}(F). \end{aligned}$$

Remark

(1) $F_{bl} = F_{n-1}(X) \backslash F_{exit}$.
(2) It could be that $x(t_1; (0, x_0)) \in F_1 \cap F_2$ with $F_1 \in F_{exit}$ and $F_2 \in F_{bl}$.

CPAHCSP.PO

Proposition. Equivalence of problem

There exists a control law for Problem control-to-exit
if and only if

1. the control law solves the Problem blocking facets with $F_{bl} = F_{n-1}(X) \backslash F_{exit}$;
2. all state trajectories of the closed-loop system leave the polytope X in finite time.

(ieeetac-v57p2831-2843, Prop. IV.7).

CPAHCSP.PO

Def. Polyhedral complex

Define a polyhedral complex $PC \subseteq P(\mathbb{R}^n)$
as a finite collection of polytopes in $\mathbb{R}^n$

1. if $P \in PC$ and if $F \in F_k(P)$ then $F \in PC$;
2. if $P, Q \in PC$ then either $P \cap Q \in (F_k(P) \cap F_k(Q))$ or $P \cap Q = \emptyset$.

$|PC| = \cup_{P \in PC} P$.
$V(PC) = \cup_{P \in PC,\ \dim(P)=0} P$, equals the vertex set of PC.

Def. Polyhedral subdivision of polytope X

PS is a polyhedral complex such that $X = |PS|$.
$QS \subseteq PS$ is a refinement if

$$((\forall\ P \in PC),\ P = \cup\{Q \in QC | Q \subseteq P\}).$$

Def. Triangulation

A triangulation $\mathrm{Tr}(X)$ of a polytope X is a polyhedral subdivision
such that for all $S \in \mathrm{Tr}(X)$, S is a simplex.

CPAHCSP.PO

Def. Compatibility of polyhedral complexes

$$\begin{aligned} & X \subset \mathbb{R}^n, \text{ full-dimensional polytope,} \\ & h : X \to \mathbb{R}^p, \ \text{ surjective function,} \\ Y &= h(X). \end{aligned}$$

Polyhedral subdivision of Y is called
(X, h)-compatible if

$$\begin{aligned} & ((\forall\, P \in PS(Y),\ \forall\, x \in V(h^{-1}(P) \cap X)),\ h(x) \in V(P)), \\ & \text{interpretation, } h \text{ maps verticies to vertices;} \\ h^{-1}(W) &= \{x \in \mathbb{R}^n | h(x) \in W\},\ W \subset \mathbb{R}^p. \end{aligned}$$

CPAHCSP.PO

Def. Chamber complex (Y, X, h)

.

$$
\begin{aligned}
& (Y, X, h),\ Y = h(X),\ h : X \to \mathbb{R}^p, \\
CHC(Y, X, h) &= \{P_y \in P(Y) | \forall\, y \in Y\}, \\
P_y &= \cap \{h(F) \in P(Y) | F \in F(X),\ y \in h(F)\}, \\
& \text{called chamber of } y \in Y.
\end{aligned}
$$

$CHC(Y, X, h)$ is a polyhedral complex.

CPAHCSP.PO

Theorem. Existence chamber complex and triangulation

(a) If the polyhedral subdivision $PS(Y)$ of Y refines the chamber complex $CHC(Y, X, h)$ then $PS(Y)$ is (X, h)-compatible.

(b) If $PC(P)$ is a polyhedral subdivision of $P \in P(\mathbb{R}^n)$ then there exists a triangulation $\mathrm{Tr}(P)$ of P that refines PC. Constructed to satisfy $V(\mathrm{Tr}) = V(PC(P))$.

(c) Consider a triangulation $\mathrm{Tr}(Y)$ of $Y = h(X)$. $\mathrm{Tr}(Y)$ is a (X, h)-compatible triangulation of Y if and only if $\mathrm{Tr}(Y)$ is a refinement of $CHC(Y, X, h)$.

CPAHCSP.PO

Procedure. Construction control law

Consider a piecewise-affine control system with output.
$\mathrm{Tr}(Y)$ a triangulation of output space Y, (X, h)-compatible.

$$((\forall\ w \in V(\mathrm{Tr}(Y))),\ \exists\ u_w \in U).$$

$$\begin{aligned}
&(1) \qquad ((\forall\ y \in Y),\ \text{choose } S_y \in \mathrm{Tr}(Y) \text{ of smallest dim. s.t. } y \in S_y);\\
&(2) \qquad y = \sum_{w \in V(S_y)} \lambda_{y,w} w,\ \ \lambda_{y,w} \in (0,1]^k,\ k = \dim(V(S_y));\\
&(3) \qquad g(y) = \sum_{w \in V(S_y)} \lambda_{y,w} u_w,\ \ g : Y \to U.
\end{aligned}$$

Then g is a continuous and piecewise-affine on $\mathrm{Tr}(Y)$ and

$$\forall\ w \in V(\mathrm{Tr}(Y)),\ g(w) = u_w.$$

CPAHCSP.PO

Theorem. Solvability problem control-to-exit

Consider an affine control system with output and,

$$
\begin{aligned}
& X \subset \mathbb{R}^n, \text{ full-dimensional polytope,} \\
F_{exit} \quad \subseteq \quad & F_{n-1}(X), \text{ set of admissable exit facets,} \\
h(x) \quad = \quad & Cx, \; h : X \to \mathbb{R}^p, \; \mathrm{rank}(C) = p; \\
& CHC(Y, X, h), \text{ chamber complex; } \; ((\forall\, w \in V(CHC)), \\
Q_w \quad = \quad & \left\{ \begin{array}{l} u \in U | \forall\, v \in V(h^{-1}(w) \cap X), \; \forall\, F \in F_{n-1}(v, X) \backslash F_{exit}), \\ h_F^T(Av + Bu + a - v) \leq 0, \end{array} \right\}.
\end{aligned}
$$

(a) Problem blocking facets is solvable
if and only if $(\forall\, w \in V(CHC(X, h)), Q_w \neq \emptyset)$.
(b) Problem control-to-exit is solvable if and only if

$$
\begin{aligned}
& ((\exists\, h_D \in \mathbb{R}^n) \text{ such that } \forall\, w \in V(CHC), \; \exists\, u_w \in V(Q_w), \\
& \forall\, v \in V(h^{-1}(w) \cap X)), \;\; h_D^T[Av + Bu_w + a - v] > 0.
\end{aligned}
$$

CPAHCSP.PO

Approach. Dynamic output feedback

Reformulate the problem of dynamic output feedback
into one with static-output feedback.

$$
\begin{aligned}
dx(t)/dt &= Ax(t) + Bu(t) + a, \; x(0) = x_0, \\
y(t) &= Cx(t), \\
dz(t)/dt &= Az(t) + Bu(t) + K[y(t) - Cz(t)], \\
u(t) &= F_1 z(t), \\
\frac{d}{dt}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} A & 0 \\ KC & A - KC \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} u(t) + \begin{pmatrix} a \\ a \end{pmatrix}, \\
\overline{y}(t) &= \begin{pmatrix} 0 & C \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \\
&\quad \text{construct static-output feedback } F_2, \\
u(t) &= F_2 \overline{y}(t) = F_2 \begin{pmatrix} 0 & C \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = F_2 C z(t) = F_1 z(t).
\end{aligned}
$$

Required to guarantee that $z(t) \in X$ for all time. Further research required.

Outline

Problems of PAHCSP

Problems of a PAHCSP

Control synthesis, realization, and
computability of piecewise-affine hybrid systems.

Remarks

- Complexity of control synthesis is the main issue for hybrid systems.
- Theory of computation and complexity for discrete sets.
  (Concept of Turing machine. Decidable and undecidable problems.)
- For real numbers, complexity theory available in books:
  (1) Blum-Cucker-Shub-Smale.
  (2) K. Weihrauch (computable analysis).
  Needed are more concepts, theorems, and experience.
- Problems of reachability and of observability of PAHSP are undecidable.
  (E.D. Sontag (1995); P.J. Collins, JHvS (CDC.2004)).
  Which problems are still decidable?
  Where is the boundary between decidable and undecidable?
  Sufficient conditions.

Problems of PAHCSP

Approaches (1). Control, realization, and computability of HS

1. Control synthesis:
   Computable sufficient conditions for existence of control laws and algorithms for control laws.
2. Realization:
   Mathematical characterization of reachable subset and of observational indistinguishability relation.
   System reduction. Existence of realizations.
   Subclasses of PAHCSP with particular properties.
3. Computability:
   For which subclass of nonlinear hybrid system is the reachable subset numerically approximable?

Problems of PAHCSP

Approaches (2). Control of PAHCSP

1. Decomposition into discrete and continuous dynamics.
2. Control at continuous level: affine control system on a polytope.
   (2.1) Control-to-a-facet.
   (2.2) Control-to-exit.
   (2.3) Stabilization-to-a-fixed-point.
3. Control at discrete level: reachability check.

Comments

Examples of PAHCSP

- Idle speed control of an automobile
  (Parades-CWI, HSCC.2004, A. Balluchi etal.)
- Control of a helicoptor
  (U. Aalborg, R. Wizniewski and students).
- Control of a two-tank system
  (TU Karlsruhe, T.E. Hodrus)

Comments

Why does there exist an explicit algebraic solution to this control problem?

- Geometry: polytopes. Convexity. Finite characterizations.
- Algebra. Affine maps.
- System. Affine dynamics in state and in input.
- Control problem. Control-to-facet problem fits the framework.
- Computability. Finite and algebraic characterizations of system and of control objective.
- System theory. Category of affine systems on polytopes (E.D. Sontag (1982)).

Discussion

Further research

- Special case. Piecewise-affine hybrid system on rectangles.
  Still many simplices.
  Approximation by one control law per rectangle.
- Control of multi-affine hybrid systems on polytopes.
  (C. Belta, L. Habets, etal.)
- Optimal control of PAHCSP?
  - Infinite-horizon optimal control?
  - Finite-horizon optimal control.
    Time-varying control law, not easy to implement.
- Constraints on the state set and on the input set.
  Handled by polytopes.

Discussion

Further research

- Robustness.
  - Control of hybrid systems on polytopes is inherently not robust.
  - Modeling may safeguard a form of robustness.
- Approximation of subclasses of nonlinear control systems on polytopes by piecewise-affine control systems on polytopes.
- Control of a PAHSP with one affine control law per polytope. Approximation!
- Abstraction of a PAHS on rectangles to an automaton. (H. de Jong etal; C. Belta etal; and D. Safranek etal.)
- Computer program ConPAHS (Margreet Nool etal.)
- Realization of linear and bilinear hybrid systems. (Mihály Petreczky etal.; not on polytopes).

Discussion

Motivation for further research

- Online control of engineering systems with computers.
- Distributed control systems
  involving interaction of discrete and continuous dynamics.

Needs of control theory of hybrid systems

- Limit the complexity of control synthesis.
- Limit the complexity of online control computations.
- Approximation of nonlinear hybrid systems on varieties
  by PAHSP or by PAHSS.
- Control synthesis of distributed hybrid systems on polytopes.

Plan for further research

- Examples of engineering problems with control of PAHSP.
- Examples of control of distributed hybrid systems.

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The end!